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MATHEMATICAL CONSIDERATIONS PERTAINING TO THE ACCURACY OF POSITION LOCATION AND NAVIGATION SYSTEMS - Part I

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13. ABSTRACT

The general case of the location of a point at the intersection of two lines-of-position involves different errors associated with each line. Also the general angle of intersection is other than 90 degrees. The distribution of error about such a point is an ellipse. A method of analysis has been developed to convert this general case to the more readily handled case of two different standard deviations along the major and minor axes of the error ellipse. Nomograms and graphs are given to facilitate computation of error probabilities. Curves are shown to indicate the geometrical magnification of error as the intersection angle departs from 90 degrees. In addition the development is shown of a total error ellipse resulting from the combination from several randomly oriented error ellipses representing error components. Numerical examples are given. Derivations of all formulas shown are given in the appendixes.

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Error Ellipse						
Probability						
Error Measure						
Combination of errors						

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Prepared for:

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WASHINGTON, D.C.
AND
THE OFFICE OF NAVAL RESEARCH
WASHINGTON, D.C.

CONTRACT Nonr-2332(00)

The methodology, views and conclusions contained herein are preliminary. Accordingly, this document does not constitute an official report of the Stanford Research Institute, and may therefore be expanded or modified at any time.

Approved: LAWRENCE J. LOW, DIRECTOR
NAVAL WARFARE RESEARCH CENTER

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PREFACE

Stanford Research Institute has been conducting a study of position location and navigation for the U.S. Marine Corps under Contract Nonr 2332(00). An important part of this study is the assessment of present and future systems. In developing criteria for such assessment, considerable difficulty was experienced in locating suitable references for the necessary mathematical analyses. Further, descriptions of systems accuracy found in the literature were found to be often confusing, sometimes ambiguous, and occasionally in error. Thus an extensive effort became necessary to collect suitable information, to develop additional methods of analysis, and to select a uniform method of specifying system accuracy.

This Research Memorandum has been prepared to record these mathematical techniques as they have been developed and used for the assessment of various position location and navigation systems. A concomitant result has been the specification of useful methods of describing the accuracy of position measurements. This memorandum follows a format of presenting the results with illustrative examples. These are followed by extended discussion and derivation of formulas in the several appendixes. The final report of the position location and navigation study is classified. Since the mathematical techniques developed for this study can be applied to other types of problems, they have been presented in a separate unclassified volume to permit wider dissemination of these techniques to those who may find them useful.

Additional volumes dealing with this subject may be issued under this and other studies as the need arises.

Typographical errors existing in the first printing of this memorandum have been corrected in this printing. No other textual changes have been made in this second printing.

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ILLUSTRATIONS

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LIST OF SYMBOLS

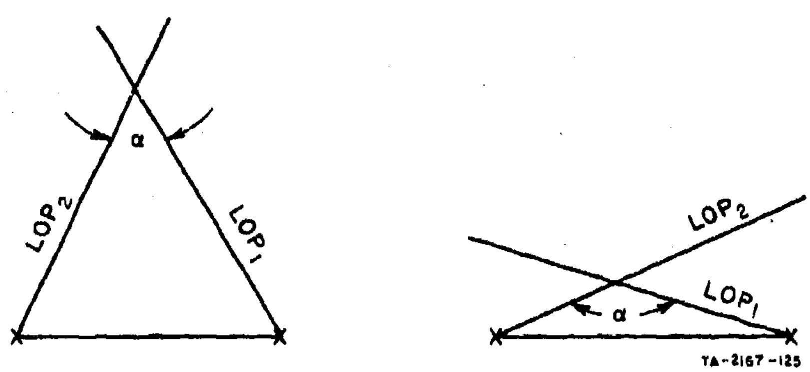
LOP Line of position

At a point determined by the intersection of two lines of position

σ_1 Standard deviation associated with LOP #1, measured perpendicular to the LOP

σ_2 Standard deviation associated with LOP #2, measured perpendicular to the LOP

α Angle between the two lines of position, also (and equivalently) angle between LOP 1 and LOP 2 as shown below.



σ^* Fictitious standard deviation

α^* Fictitious intersection angle
The combination of σ^*, σ^*, and α^* represents an equivalent description of a probability distribution actually described by σ_1, σ_2, and α.

After transformation of a probability distribution described by σ_1, σ_2, and α to an equivalent distribution defined in terms of the axes of an ellipse

σ_x standard deviation along major axis

σ_y standard deviation along minor axis

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When distributions about several points are combined to obtain an overall distribution, double subscripts are used to designate the separate ellipses which describe the distributions about each point.

σ_{11}	Standard deviation LOP #1—ellipse #1
σ_{21}	Standard deviation LOP #2—ellipse #1
σ_{12}	Standard deviation LOP #1—ellipse #2, etc.

After transformation to standard deviations along the axes of the individual ellipses.

σ_{x1}	Standard deviation along major axis—ellipse #1
σ_{y1}	Standard deviation along minor axis—ellipse #1
σ_{x2}	Standard deviation along major axis—ellipse #2, etc.
θ_1	Angle between x-axis of ellipse #1 and arbitrary coordinate axes for combination of ellipses
σ_{w1}, σ_{z1} σ_{w2}, σ_{w2} etc.	Standard deviations along the arbitrarily selected axes, designated w and z, for multiple ellipse combination.
ρ_1, ρ_2, etc.	Cross product function involved with transformation to w and z axes.

In general discussion of several ellipses the subscript i is used to designate the general function. f = 1,2,3,...........

In the final combined ellipse, the subscript f is used.

σ_{wf} σ_{zf} ρ_f	As above for the final ellipse
σ_{xf}	Standard deviation along major axis
σ_{yf}	Standard deviation along minor axis
θ_f	Angle between x-axis and arbitrary coordinate axes.

1. BASIC STATEMENT OF POSITION LOCATION PROBLEM

The basic problem in position location is the determination of the coordinates of a remote point with respect to a known or arbitrary reference. The remote point may be a landing zone for troops, or a target upon which it is desired to deliver ordnance. Many other examples will immediately come to mind. The problem giving rise to the necessity for complex mathematical analysis is the fact that no measurement can be made without error. Thus, the results of a position determination in fact must be described in terms of the probability of being within a given distance of the desired point. Actually, this last statement is in too simple terms: because more than one error is usually involved in the sum total of measments, it becomes necessary to consider the shape of the probability distribution about the desired point. In general, these probability distributions are ellipses rather than circles.

A relatively simple problem in position location is given in Fig. 1. Here is assumed a reference baseline established by the measurement system. The location of an artillery battery is then measured by the system, giving rise to an ellipse within which a given probability may be stated that the artillery battery is actually located. Then, with respect to the same baseline, a forward observer is located, giving rise to a second ellipse within which he may be located to a stated percentage probability. From his location he makes measurements on a target which then may be located within a still different ellipse. From this information the dotted line giving firing orders in range and azimuth is calculated. The weapon effects ellipse is shown dotted and superimposed upon the target location ellipse. Then, the problem of immediate interest is to calculate the probability of damage to the target. Techniques have been established to perform such calculation when each of the error figures about the various points in the problem is a circle. However, the use of such circles can be quite misleading when the actual figures are ellipses. In particular, the weapons effect pattern is commonly a very elongated ellipse, differing greatly from a circle. It is also characteristic of many electronic measuring

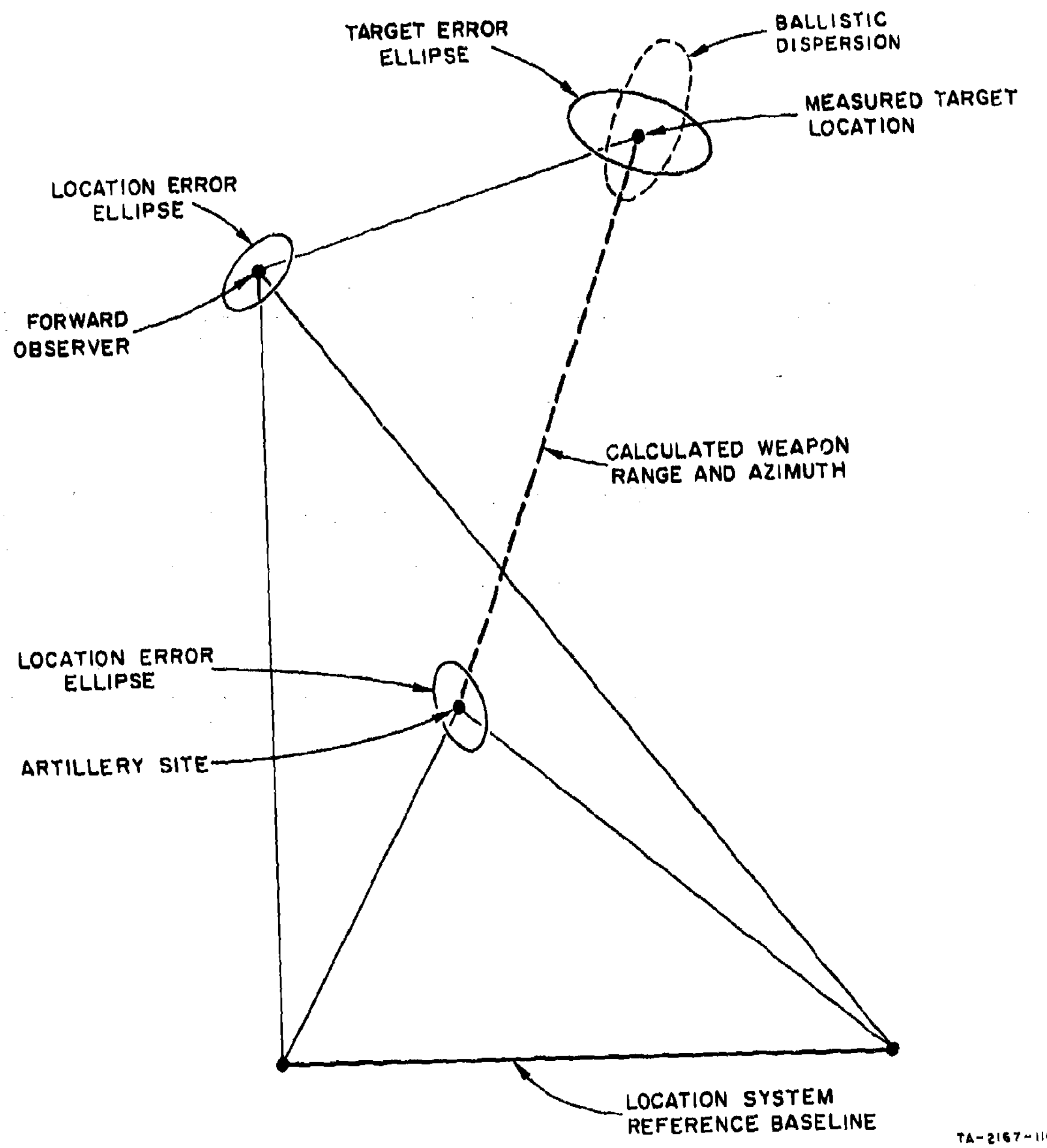

FIG. 1 BASIC LOCATION PROBLEM

techniques that the results, to be meaningful, must be expressed in terms of ellipses. Thus it becomes necessary to develop a procedure of analysis which permits the consideration of ellipses.

A circle is readily defined in terms of its center and a single distance, the radius. An ellipse requires additional information—the center, and two distances to correspond to the radius of a circle—the semi-major axis and the semi-minor axis. Further, when we are concerned with more than one ellipse, we must also be concerned with their relative orientations. Thus, any analytical procedure concerned with the end result of the consideration of a number of error ellipses considers the elements of each ellipse and the angles of these elements with respect to a common reference.

Because the detailed consideration of analytical techniques concerned with any single ellipse is itself quite complex, the discussion will detail first the considerations of a single ellipse of error. Following this presentation, the method of combining several ellipses to obtain the end result will be described.

2. ANALYSIS OF A SINGLE ERROR ELLIPSE

Most of the position location systems considered in this program determine the location of a point at the intersection of two lines of position. However, both lines of position may be in error. Figure 2 shows such an intersection of two lines of position. The lines of position in this illustration are range measurements from two points at the extremities of a baseline of known length. The measurements of ranges are reported as some numerical value. However, because of inaccuracies in measurement, the actual range may not be the indicated value, but may lie somewhere between the limits shown as additional arcs either side of the measured line. Thus, one becomes interested in the probability that the actual point lies within some close distance of the indicated point.

The intersection of the two lines of position together with the standard deviations associated with each are shown to expanded scale in Fig. 3. (Standard deviation is a measure of error and is defined in Appendix A.) Standard deviation as a measure of error is commonly designated by the Greek letter sigma (σ) and the Greek letter alpha (α) will be used throughout the analysis to designate the angle of intersection of two lines of position. It can be shown that the contours of equal probability density about such an intersection are ellipses centered about the intersection of the two lines of position. Thus, the ellipse shown in Fig. 3 might be the 75% probability ellipse, meaning that there is a 75% probability—three chances in four—that the actual position of the point whose location is desired lies within the ellipse drawn.

The detailed statistical analysis of the diagram shown in this figure is quite complex. Salient features of the analysis will be stated in the main portion of this memorandum. Rigorous mathematical analyses will be found in the appendixes. This main body of the memorandum will indicate the assumptions made in the analysis of the accuracy of position location systems and will present formulas, graphs, and nomograms to obtain numerical results. Typical examples will be illustrated.

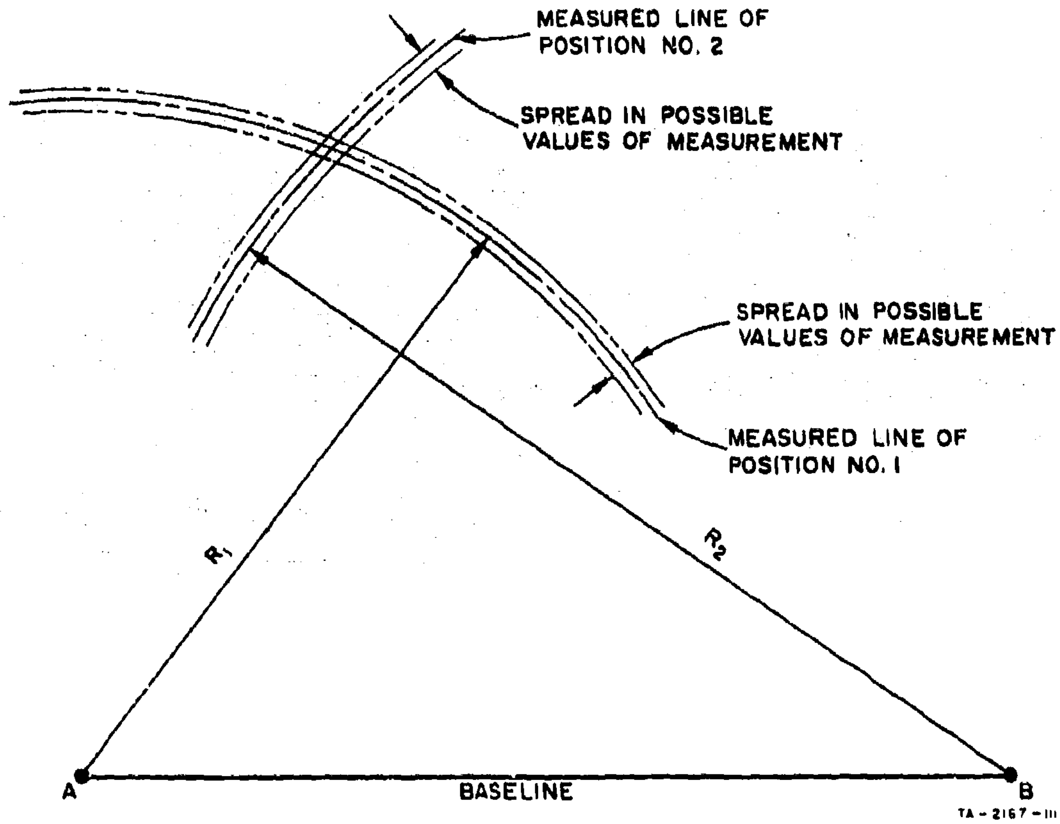

FIG. 2 POSITION LOCATION AT INTERSECTION OF TWO LINES OF POSITION

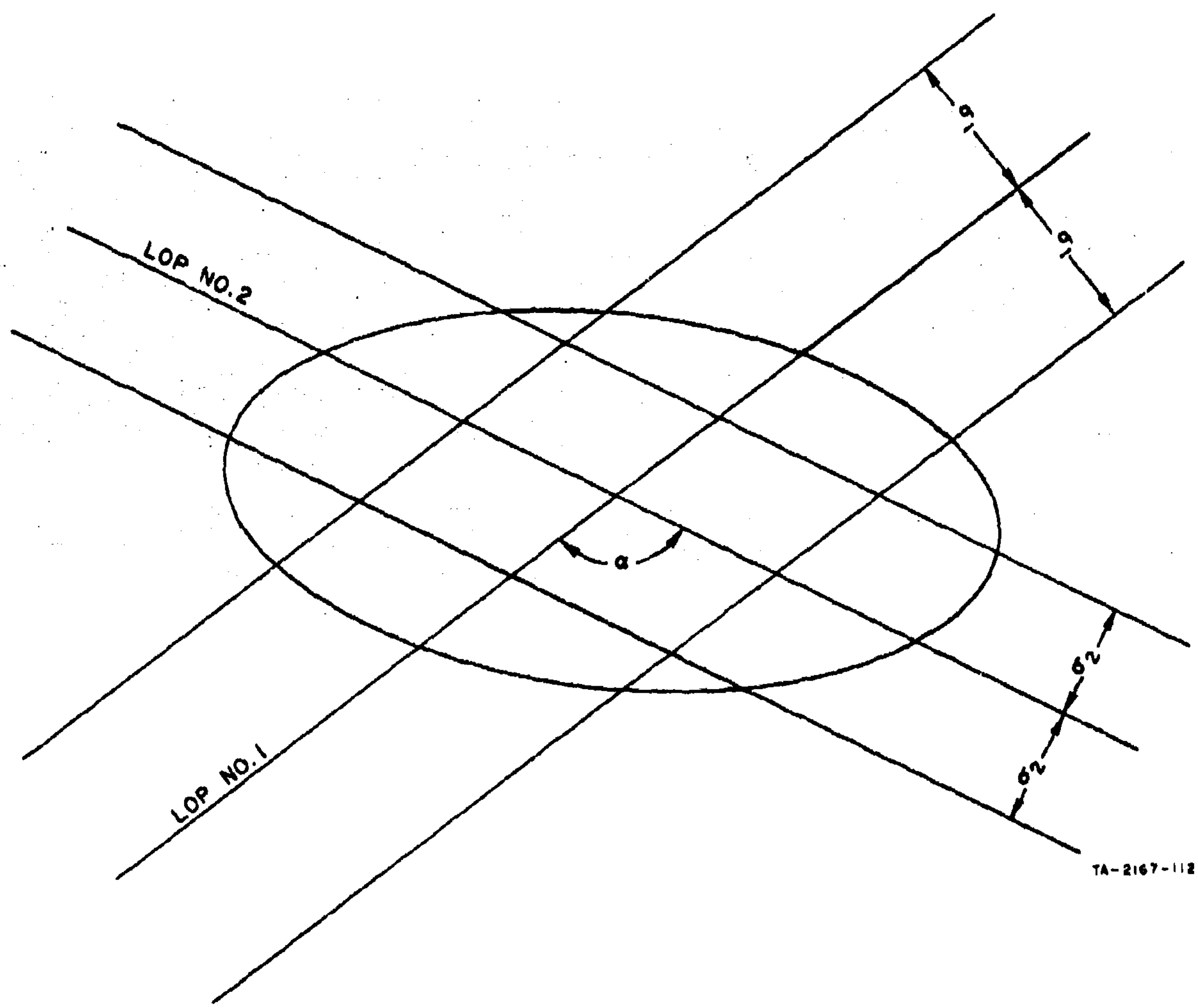

FIG. 3 EXPANDED VIEW OF INTERSECTION OF TWO LINES OF POSITION

In the analysis to follow of the accuracy of position location systems the following assumptions have been made:

(1) All bias errors have been removed, leaving only the random errors to be analyzed. In mathematical terms, the mean or average error is assumed to be zero.

(2) These random errors are assumed to be normally distributed. The mathematical implications of this assumption are discussed in following paragraphs. This assumption is required to permit a mathematical statement of overall accuracy relationships to be developed.

(3) The errors associated with the two lines of position are assumed to be independent. This assumption implies that a change in the error of one line of position has no effect upon the other. This assumption permits the analysis to consider unequal errors associated with the two lines of position, thus assuring a realistic mathematical model.

(4) The lines of position are assumed to be straight lines over the small area of interest in the neighborhood of the point, the position of which is desired. (See Fig. 3.) This assumption is valid so long as the standard deviation is small with respect to the actual radius of curvature of the line of position. In the analysis of systems this is usually found to be the case. Not to make this assumption would unreasonably complicate the mathematical analysis.

(5) The analysis of errors of position is confined to two dimensions. The third dimension, altitude, may be considered separately, if desired, if the system being analyzed is capable of altitude measurements.

As shown in Fig. 3, the general case of the intersection of two lines of position at any angle and with different values of error associated with each line of position results in an elliptical error figure. Simplified to geometrical terms, the ellipse looks like that of Fig. 4.

From this illustration, one may readily surmise that the exact shape of the error figure varies with the magnitudes of the two input errors, σ_1 and σ_2, as well as with the intersection angle, α. The angle α is also the angle between the two values of sigma because the standard deviations are mutually perpendicular to their corresponding lines of position. How these variations may be calculated was the objective of considerable analytical effort by the project team after a literature survey indicated the inadequacy of available techniques.

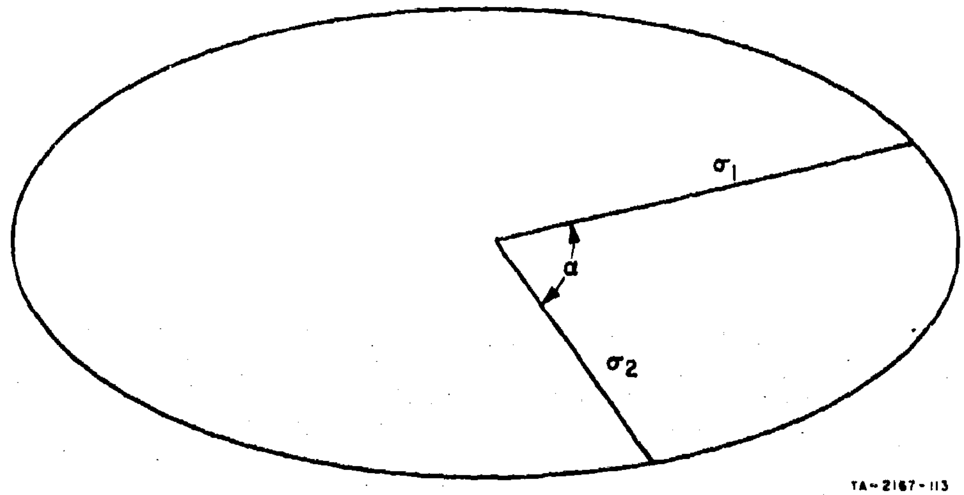

FIG. 4 BASIC ERROR ELLIPSE

Results of this analytical effort have given two separate but interrelated methods of obtaining the desired results. Each will be described as simply as possible and the necessary formulas given with examples. Detailed deviations of the formulas obtained may be found in the appendixes.

Using either of the two methods of analysis developed, the end result is the determination of the probability that the point is located within a circle of stated radius. The basis of this concept may best be seen by considering for a moment the special case when the two errors are equal and the angle of intersection of the lines of position is a right angle. In this case, and in this case alone, the error figure becomes a circle and is described by the circular normal distribution. A plot of this special function is given in Fig. 5. The plot is to be interpreted as follows: the horizontal axis is measured in terms of R/σ, R being the radius of a circle within which it is desired to be located, and σ being the error measure. The error measure is given simply as σ, for in this circular case $\sigma_1 = \sigma_2$. To illustrate, a measurement system gives a circular error figure and has a value of $\sigma = 10$ meters; the probability of actually being located within a circle of 10 meters radius when $R/\sigma = 1.0$ may be read from the vertical axis to be 39.3%. To obtain the CEP, the radius of a circle within which a 50% probability results, the corresponding value of R/σ is seen to be 1.18 from the graph. Thus, for this example, the CEP would be 11.8 m. The concept of the function R/σ will be found useful in more complex cases.

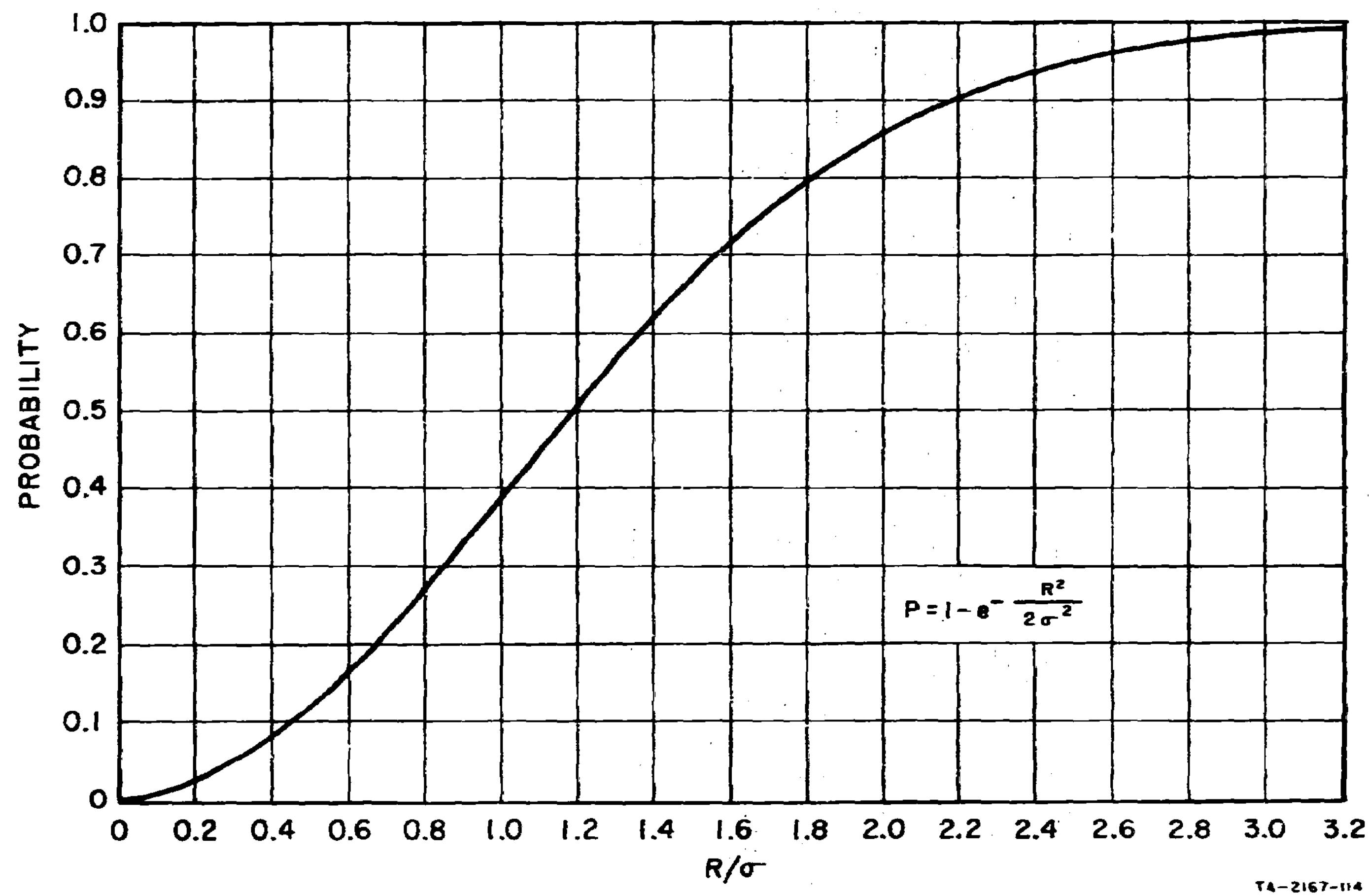

FIG. 5 CIRCULAR NORMAL DISTRIBUTION

2.1 METHOD 1

Starting with the inputs of Fig. 4, it is assumed that it is possible to find fictitious values of sigma so that the two differing values originally given may be replaced by two new sigmas of identical value, indicated as σ^*. At the same time a new and fictitious angle of intersection α^* is also required. Figure 6 indicates these new values.

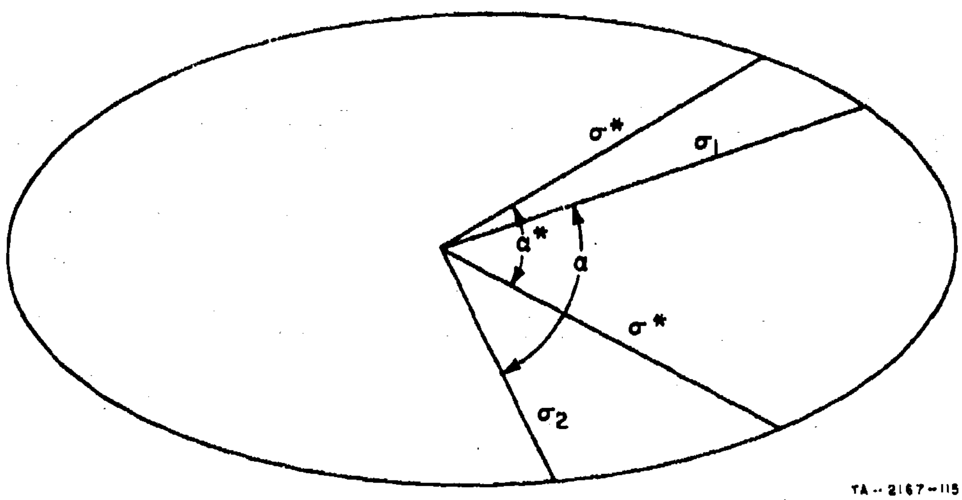

FIG. 6 TRANSFORMED PARAMETERS OF ERROR ELLIPSE

Also required for use with this first method is a whole set of probability curves, similar to that of Fig. 5, but with a separate curve for each value of intersection angle. Such curves have been calculated with the aid of a digital computer and are shown on Fig. 7. These curves can be used only when the two error measures are equal, hence the need for making the transformation of the previous paragraph.

The values of the functions σ^* and α^* needed to utilize the curves of Fig. 7 may either be computed from the formulas given below or may be determined from Figs. 8 and 9.

$$\sigma^* = \frac{\sin 2\beta\sqrt{\sigma_1^2 + \sigma_2^2}}{\sqrt{2}}$$

$$\alpha^* = \arcsin (\sin 2\beta \sin \alpha)$$

where

$$\beta = \arctan (\sigma_1/\sigma_2)$$

thus

$$\sin 2\beta = \frac{2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2} \quad .$$

The derivation of these formulas may be found in Appendix D. It is obvious that considerable computation is required to use the formulas directly, hence the curve of Fig. 8 and the nomogram of Fig. 9 will be found to facilitate the use of the formulas.

First one must calculate the ratio σ_2/σ_1. σ_1 is always taken as the larger of the two in this fraction, such that the value is always less than 1.0. With this ratio, enter the curve of Fig. 8 and obtain the σ^* factor. Multiply σ_1 by this factor to obtain the fictitious function σ^*. The nomogram of Fig. 9 is used with the same ratio to obtain the fictitious angle α^*.

A numerical example will illustrate the method of calculation.

Assume a position location system has provided the following data:

$$\alpha = 50^\circ$$
$$\sigma_1 = 20 \text{ meters}$$
$$\sigma_2 = 15 \text{ meters} \quad .$$

What is the probability of the location of the point within a circle of 30 meters radius?

Calculate the ratio $\sigma_2/\sigma_1 = 15/20 = 0.75$.

Enter the curve of Fig. 8 with this value and obtain the σ^* factor = 0.845. Multiply this value by $\sigma_1 = 20$ to obtain $\sigma^* = 16.9$ meters. Calculate the ratio $R/\sigma^* = 30/16.9 = 1.78$.

Enter the nomogram of Fig. 9 with the same ratio $\sigma_2/\sigma_1 = 0.75$ and with the given angle $\alpha = 50^\circ$ to obtain the fictitious angle $\alpha^* = 47^\circ$.

The values $R/\sigma^* = 1.78$ and $\alpha^* = 47^\circ$ may be entered on the curves of Fig. 7 to obtain $P = 0.62$ or 62%, interpolating visually between the 40° and 50° curves for $\alpha^* = 47^\circ$.

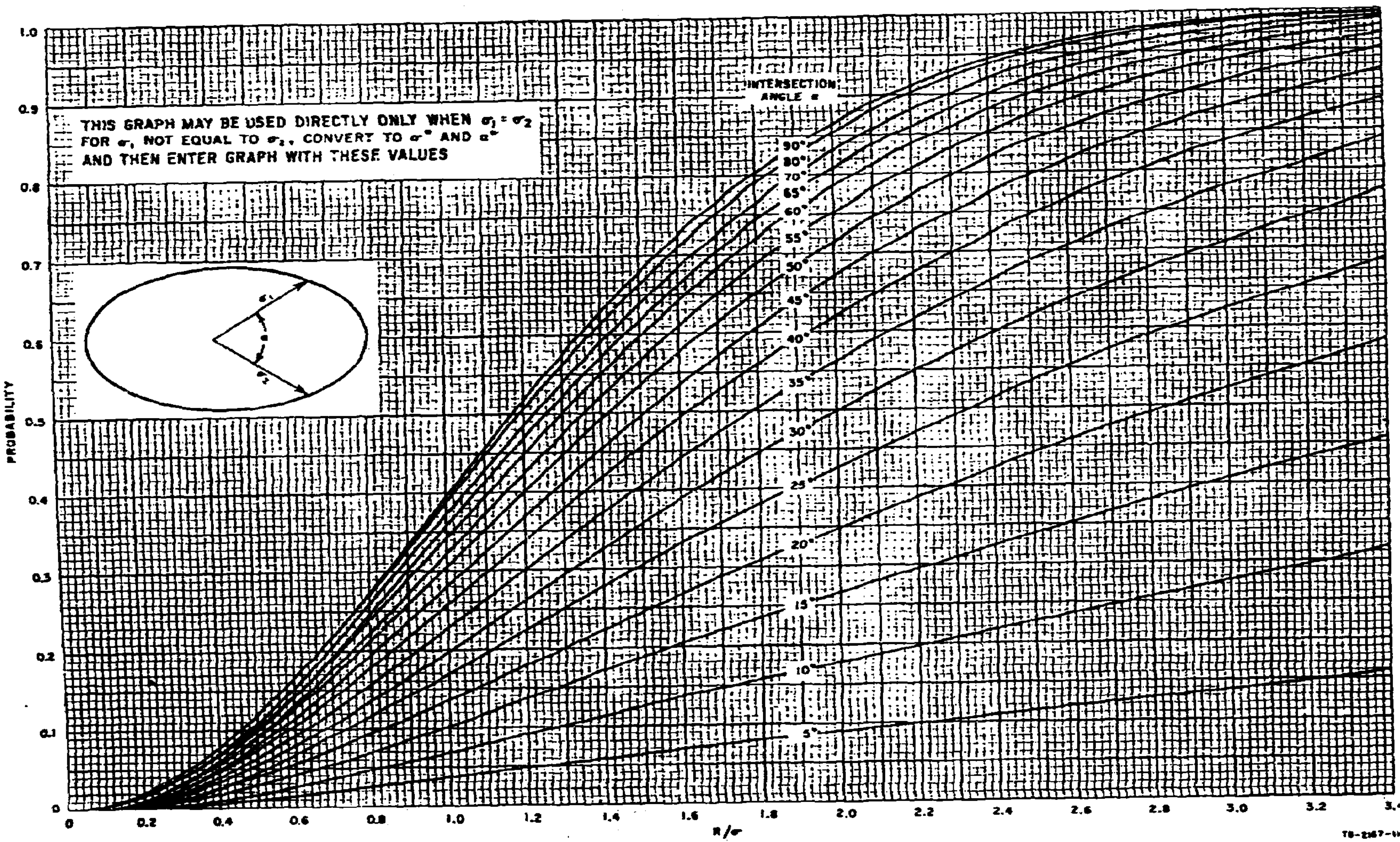

FIG. 7 PROBABILITY vs. R/σ AND α
FOR ELLIPTICAL BIVARIATE DISTRIBUTIONS
WITH TWO EQUAL STANDARD DEVIATIONS

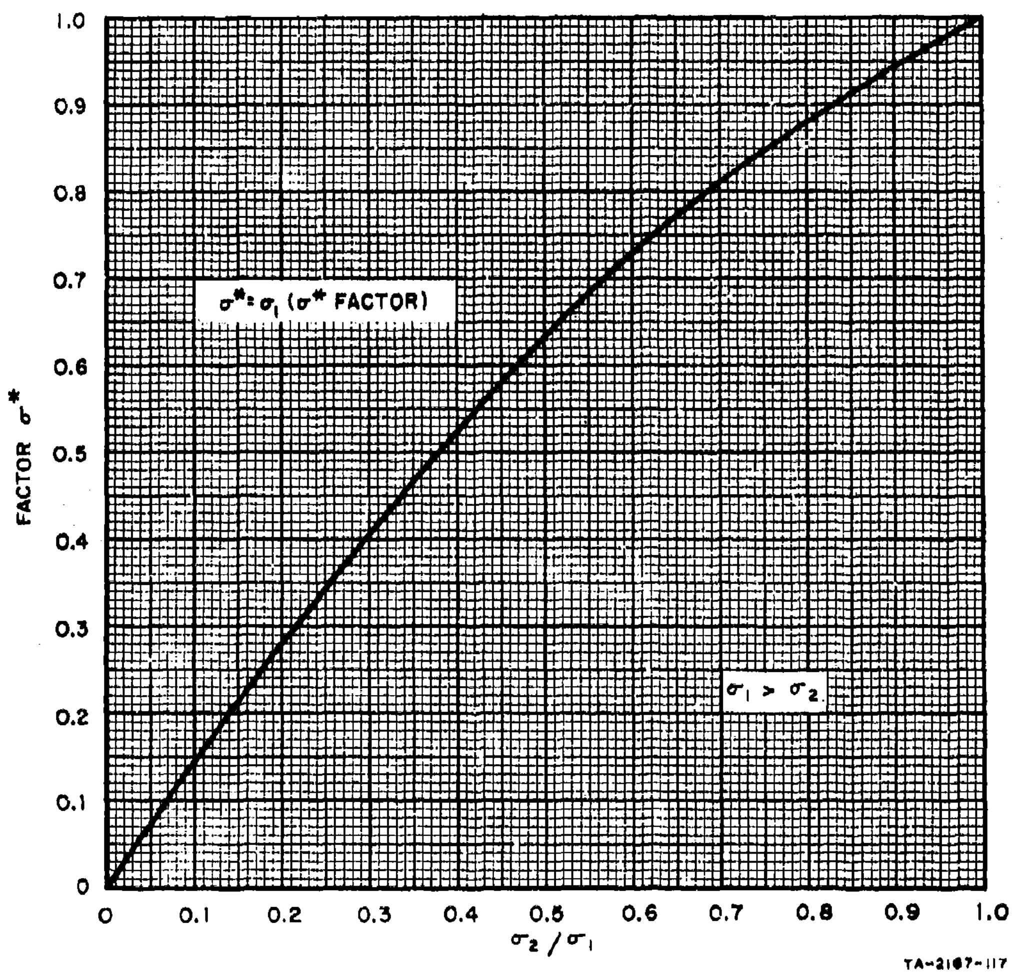

FIG. 8 σ^* FACTOR vs. σ_2/σ_1 RATIO

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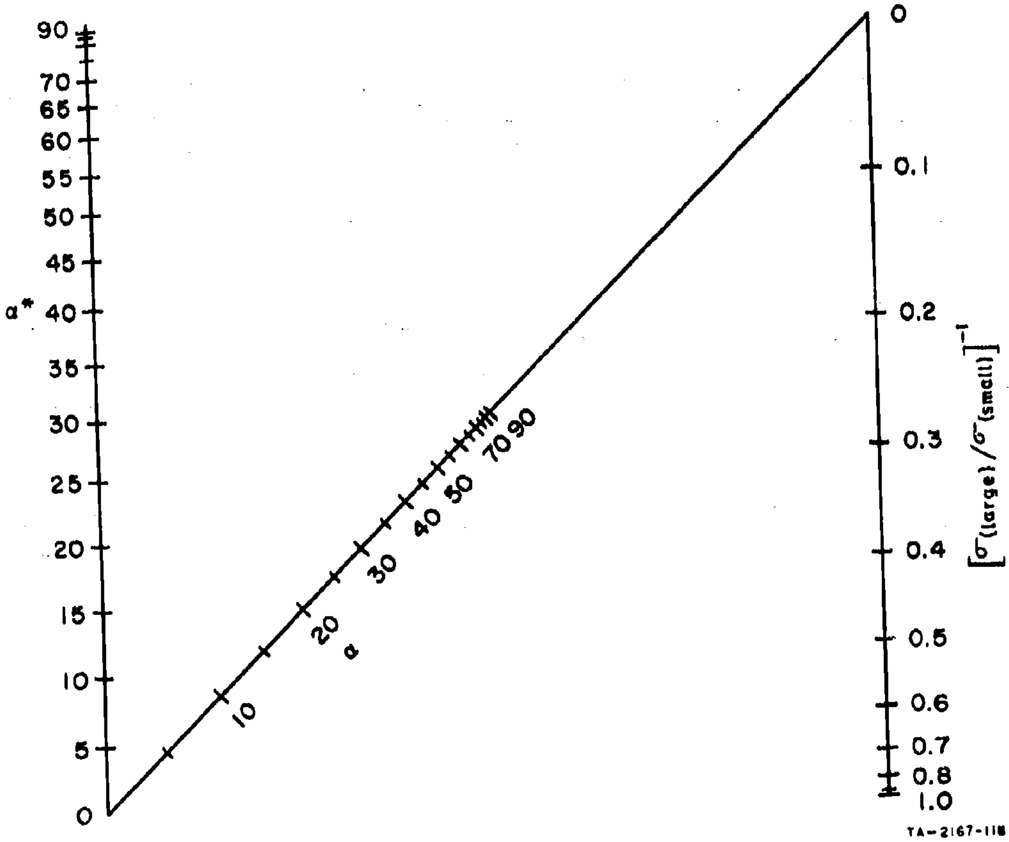

FIG. 9 NOMOGRAM TO OBTAIN a^*

Checking the values obtained from the nomograms by substituting the figures into the exact formulas we obtain:

$$\sin 2\beta = \frac{2 \times 15 \times 20}{225 + 400} = \frac{600}{625} = 0.96$$

$$\sin 50° = 0.766$$

$$\alpha^* = \arcsin (0.96 \times 0.766)$$

$$= \arcsin 0.735$$

$$= 47.3° \text{ (compared with } 47° \text{ from the graphical solution)}$$

$$\sigma^* = \frac{0.96\sqrt{625}}{\sqrt{2}}$$

$$= 0.707 \times 0.96 \times 25$$

$$= 16.96$$

$$R/\sigma^* = 30/16.96 = 1.77 \text{ (compared with 1.8 from the graphical solution)}$$

The values obtained from the figures are seen to be sufficiently accurate for comparative evaluation.

Because the curves of Fig. 7 tend to crowd quite closely together for some values when the angles approach 90 degrees, Table I presents the same data in numerical form.

2.2 METHOD 2

A second method of working with the error ellipses starts with a different transformation. In this method new values of sigma are found along the major and minor axes of the ellipse according to the formulas given below. The geometrical considerations are given in Fig. 10.

Table I

$P(R/\sigma, \alpha)$ CIRCULAR ERROR PROBABILITIES

R/σ \ α	5°	10°	15°	20°	25°	30°	35°	40°	45°	50°	55°	60°	65°	70°	75°	80°	85°	90°
0.0																		.000
0.1																		.005
0.2	.001	.004	.006	.007	.009	.011	.011	.012	.014	.015	.016	.017	.018	.018	.019	.019	.019	.020
0.3	.005	.009	.012	.015	.019	.022	.025	.028	.031	.034	.036	.038	.040	.041	.042	.043	.044	.044
0.4	.008	.014	.021	.027	.033	.039	.045	.050	.055	.059	.063	.067	.070	.072	.074	.076	.077	.077
0.5	.009	.019	.029	.039	.048	.057	.066	.074	.083	.090	.096	.102	.106	.110	.114	.116	.117	.118
0.6	.014	.029	.043	.057	.070	.082	.095	.106	.117	.127	.135	.143	.150	.156	.159	.162	.164	.165
0.7	.020	.039	.057	.075	.093	.110	.126	.141	.155	.168	.179	.189	.198	.205	.211	.215	.217	.217
0.8	.025	.048	.072	.095	.117	.138	.158	.177	.195	.211	.226	.238	.249	.258	.265	.270	.273	.274
0.9	.031	.060	.088	.116	.143	.169	.196	.217	.238	.257	.275	.290	.303	.314	.322	.328	.332	.333
1.0	.035	.070	.104	.137	.169	.199	.228	.256	.281	.305	.326	.343	.358	.371	.381	.388	.392	.393
1.1	.040	.080	.120	.158	.196	.231	.265	.296	.325	.352	.366	.396	.414	.428	.440	.448	.452	.454
1.2	.046	.092	.136	.181	.224	.264	.302	.337	.370	.400	.426	.449	.469	.485	.497	.506	.512	.513
1.3	.053	.104	.155	.204	.251	.296	.339	.378	.414	.446	.475	.501	.522	.540	.553	.563	.569	.570
1.4	.058	.115	.171	.226	.278	.327	.374	.417	.456	.491	.523	.550	.573	.582	.606	.617	.623	.625
1.5	.063	.126	.188	.247	.304	.358	.408	.455	.497	.535	.568	.597	.621	.641	.656	.667	.673	.675
1.6	.069	.138	.204	.269	.330	.388	.442	.491	.536	.576	.611	.640	.666	.686	.702	.713	.720	.722
1.7	.074	.148	.220	.289	.355	.417	.471	.526	.573	.615	.650	.682	.707	.728	.744	.756	.762	.764
1.8	.079	.159	.236	.310	.380	.445	.505	.559	.608	.651	.688	.779	.745	.766	.782	.793	.800	.802
1.9	.084	.169	.251	.329	.403	.471	.536	.591	.641	.685	.722	.754	.780	.800	.816	.827	.833	.836
2.0	.089	.179	.266	.349	.426	.498	.563	.621	.672	.716	.753	.785	.810	.832	.846	.856	.862	.865
2.1	.093	.189	.280	.367	.448	.523	.589	.649	.700	.744	.780	.812	.837	.857	.871	.881	.887	.890
2.2	.099	.199	.295	.386	.470	.546	.615	.675	.727	.771	.808	.838	.861	.880	.894	.903	.909	.911
2.3	.104	.209	.309	.404	.490	.570	.639	.700	.752	.795	.831	.860	.883	.900	.913	.922	.927	.929
2.4	.109	.219	.324	.422	.512	.592	.663	.724	.775	.818	.852	.880	.901	.918	.930	.938	.942	.944
2.5	.115	.229	.338	.440	.532	.614	.686	.746	.797	.838	.871	.897	.918	.933	.943	.951	.955	.956
2.6	.119	.239	.352	.457	.551	.634	.706	.766	.816	.856	.888	.912	.931	.945	.954	.961	.964	.966
2.7	.124	.248	.366	.474	.570	.654	.726	.786	.835	.873	.903	.926	.943	.955	.964	.969	.973	.974
2.8	.129	.258	.379	.490	.589	.674	.746	.805	.851	.888	.916	.937	.953	.963	.971	.976	.979	.980
2.9	.134	.268	.393	.507	.607	.692	.763	.821	.867	.902	.928	.947	.961	.971	.978	.982	.984	.985
3.0	.139	.277	.406	.522	.624	.710	.781	.837	.880	.914	.938	.956	.968	.976	.982	.986	.988	.989
3.1	.144	.287	.419	.538	.641	.727	.797	.852	.893	.924	.947	.963	.974	.982	.986	.989	.991	.992
3.2	.149	.296	.432	.553	.657	.743	.812	.865	.905	.934	.955	.969	.979	.985	.990	.992	.993	.994
3.3	.154	.306	.445	.568	.673	.759	.826	.878	.916	.943	.962	.975	.983	.989	.992	.994	.995	.996
3.4	.159	.315	.458	.583	.688	.773	.840	.889	.925	.951	.968	.979	.986	.991	.994	.996	.997	.997
3.5	.164	.324	.470	.597	.703	.788	.852	.900	.934	.957	.973	.983	.989	.993	.996	.997	.998	.998
3.6	.169	.333	.482	.611	.717	.801	.864	.910	.942	.963	.977	.986	.991	.995	.997	.998	.998	.999
3.7	.174	.343	.494	.625	.731	.814	.875	.919	.949	.968	.981	.990	.993	.996	.998	.998	.999	.999
3.8	.179	.352	.506	.638	.744	.826	.886	.927	.955	.973	.984	.990	.995	.997	.998	.999	.999	.999
3.9	.183	.360	.518	.651	.757	.838	.895	.935	.961	.977	.987	.993	.996	.998	.999	.999	.999	1.000
4.0	.189	.370	.529	.664	.770	.848	.904	.942	.966	.980	.989	.994	.997	.998	.999	.999	1.000	1.000
4.1	.193	.378	.541	.676	.781	.859	.913	.948	.970	.983	.991	.995	.998	.999	.999	1.000	1.000	1.000
4.2	.198	.387	.552	.688	.793	.869	.920	.954	.974	.986	.993	.996	.998	.999	.999	1.000	1.000	1.000
4.3	.203	.396	.564	.700	.804	.878	.927	.958	.978	.988	.994	.997	.999	.999	1.000	1.000	1.000	1.000
4.4																		1.000

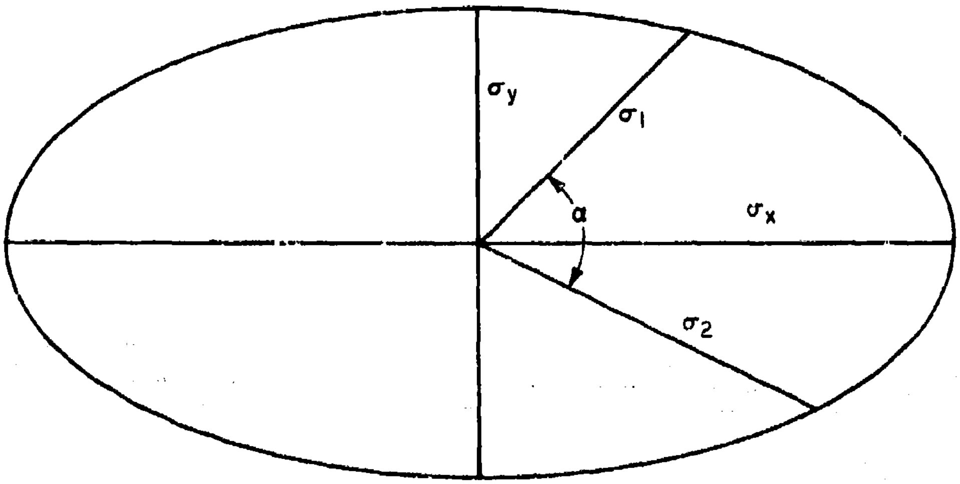

FIG. 10 TRANSFORMATION TO STANDARD DEVIATIONS ALONG ELLIPSE AXES

$$\sigma_x^2 = \frac{1}{2\sin^2\alpha}\left[\sigma_1^2 + \sigma_2^2 + \sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4\sin^2\alpha\,\sigma_1^2\sigma_2^2}\right]$$

$$\sigma_y^2 = \frac{1}{2\sin^2\alpha}\left[\sigma_1^2 + \sigma_2^2 - \sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4\sin^2\alpha\,\sigma_1^2\sigma_2^2}\right] .$$

The complex derivation of these formulas is given in its entirety in Appendix B. Note that these formulas are given terms of variances—squares of the standard deviations.

After the values of sigma along the orthogonal axes of the ellipse have been obtained, the results of computations obtained by Harter (Ref. 4) may be utilized to obtain desired circles of probability. To utilize Harter's data it is first necessary to compute the ratio $c = \sigma_y/\sigma_x$ where σ_x is the larger of the two new standard deviations just computed. Harter presents tables (Tables II and III) which then relate ellipses of varying values of ellipticity to the radii of circles of equivalent probability.

The use of this second method will be shown by using the same example given as an illustration for the first method.

Table II

CIRCULAR ERROR PROBABILITIES $P(K, c)$

K \ c	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	.0796557	.0443987	.0242110	.0164170	.0123875	.0099377	.0082940	.0071157	.0062299	.0055400	.0049875
0.2	.1585194	.1339783	.0884533	.0628306	.0482413	.0390193	.0327123	.0281416	.0246824	.0219757	.0198013
0.3	.2358228	.2213804	.1739300	.1318281	.1039193	.0851635	.0719102	.0621380	.0546598	.0487639	.0440025
0.4	.3108435	.3010228	.2635181	.2130084	.1742045	.1451808	.1237982	.1076237	.0950495	.0850326	.0768837
0.5	.3829249	.3755884	.3481790	.3003001	.2532953	.2152886	.1857448	.1626829	.1443941	.1296286	.1175031
0.6	.4514938	.4457708	.4255005	.3846374	.3357384	.2914682	.2548177	.2251114	.2009707	.1811783	.1647298
0.7	.5160727	.5115048	.4960683	.4633258	.4170802	.3699305	.3280302	.2925054	.2629373	.2381583	.2172055
0.8	.5762892	.5725957	.5604457	.5349387	.4941882	.4474207	.4025628	.3627122	.3283453	.2989700	.2738510
0.9	.6318797	.6288721	.6191254	.5993140	.5651564	.5213998	.4759375	.4333628	.3953279	.3620135	.3330232
1.0	.6826895	.6802325	.6723588	.6568242	.6291249	.5900953	.5461319	.5025790	.4621421	.4257553	.3934693
1.1	.7286679	.7266507	.7202682	.7079081	.6859367	.6524489	.6116316	.5687407	.5272462	.4887873	.4539256
1.2	.7698607	.7682215	.7630305	.7532175	.7359558	.7079073	.6714260	.6306168	.5893404	.5498730	.5132477
1.3	.8063990	.8050648	.8008554	.7929968	.7793550	.7567265	.7249673	.6873122	.6474394	.6079822	.5704426
1.4	.8384867	.8374049	.8340018	.8277048	.8169851	.7989288	.7720889	.7383089	.7007900	.6623035	.6240889
1.5	.8663856	.8655127	.8627728	.8577862	.8493071	.8350816	.8129287	.7833962	.7489500	.7122546	.6753475
1.6	.8904014	.8897008	.8875060	.8834914	.8768844	.8657559	.8478493	.8226246	.7917194	.7574708	.7219627
1.7	.9108691	.9103102	.9085619	.9053766	.9001746	.8915536	.8773110	.8562471	.8291137	.7977882	.7462539
1.8	.9281394	.9276964	.9263125	.9237989	.9197275	.9130680	.9019110	.8846624	.8613238	.8332175	.8021013
1.9	.9425669	.9422182	.9411290	.9391588	.9359855	.9303615	.9222277	.9083609	.8886731	.8639149	.8355255
2.0	.9544997	.9542272	.9533775	.9518415	.9493815	.9454540	.9388418	.9278709	.9115762	.8901495	.8646647
2.1	.9642712	.9640508	.9634011	.9622127	.9603170	.9573205	.9522999	.9437668	.9305013	.9122714	.8897495
2.2	.9721931	.9720304	.9715237	.9706109	.9691597	.9668845	.9631017	.9565522	.9459386	.9306821	.9110784
2.3	.9785518	.9784275	.9780408	.9773450	.9762419	.9745230	.9716034	.9667300	.9583739	.9458085	.9289946
2.4	.9836049	.9835108	.9832180	.9826918	.9818594	.9805703	.9784061	.9747495	.9682698	.9580804	.9438652
2.5	.9875807	.9875100	.9872900	.9868953	.9862720	.9853112	.9837569	.9810035	.9760522	.9679136	.9560631
2.6	.9906776	.9906249	.9904612	.9901674	.9897045	.9889934	.9878527	.9858331	.9821023	.9756969	.9659525
2.7	.9930661	.9930271	.9929062	.9926894	.9923483	.9918260	.9909944	.9895268	.9867530	.9817837	.9738786
2.8	.9948897	.9948612	.9947727	.9946141	.9943649	.9939842	.9933821	.9923249	.9902888	.9864876	.9801589
2.9	.9962684	.9962477	.9961834	.9960684	.9958878	.9956126	.9951798	.9944246	.9929472	.9900803	.9850792
3.0	.9973002	.9972853	.9972391	.9971564	.9970266	.9968294	.9965205	.9959554	.9949774	.9927925	.9888910
3.1	.9980648	.9980542	.9980212	.9979622	.9978699	.9977296	.9975109	.9971348	.9963851	.9948168	.9918113
3.2	.9986257	.9986182	.9985949	.9985533	.9984880	.9983892	.9982356	.9979733	.9974478	.9963105	.9940240
3.3	.9990332	.9990279	.9990116	.9989824	.9989368	.9988677	.9987607	.9985792	.9982147	.9974004	.9956822
3.4	.9993261	.9993225	.9993112	.9992909	.9992593	.9992115	.9991376	.9990129	.9987626	.9981868	.9969113
3.5	.9995347	.9995323	.9995245	.9995105	.9994888	.9994559	.9994053	.9993204	.9991502	.9987480	.9978125
3.6	.9996818	.9996801	.9996748	.9996653	.9996505	.9996281	.9995938	.9995364	.9994218	.9991442	.9984662
3.7	.9997844	.9997832	.9997797	.9997733	.9997633	.9997482	.9997251	.9996867	.9996102	.9994208	.9989352
3.8	.9998553	.9998545	.9998522	.9998478	.9998412	.9998311	.9998157	.9997902	.9997396	.9996119	.9992682
3.9	.9999038	.9999033	.9999018	.9998989	.9998945	.9998876	.9998776	.9998608	.9998276	.9997420	.9995020
4.0	.9999367	.9999363	.9999353	.9999334	.9999305	.9999261	.9999195	.9999085	.9998870	.9998309	.9996645
4.1	.9999587	.9999585	.9999578	.9999566	.9999547	.9999519	.9999475	.9999404	.9999266	.9998900	.9997763
4.2	.9999733	.9999732	.9999727	.9999720	.9999707	.9999689	.9999661	.9999616	.9999527	.9999292	.9998523
4.3	.9999829	.9999828	.9999826	.9999821	.9999813	.9999801	.9999783	.9999754	.9999698	.9999548	.9999034
4.4	.9999892	.9999891	.9999889	.9999886	.9999881	.9999874	.9999863	.9999845	.9999809	.9999715	.9999375
4.5	.9999932	.9999932	.9999931	.9999929	.9999925	.9999921	.9999914	.9999902	.9999881	.9999822	.9999599
4.6	.9999958	.9999957	.9999957	.9999955	.9999954	.9999951	.9999947	.9999939	.9999926	.9999889	.9999746
4.7	.9999974	.9999974	.9999973	.9999973	.9999971	.9999970	.9999967	.9999963	.9999955	.9999932	.9999840
4.8	.9999984	.9999984	.9999984	.9999983	.9999983	.9999982	.9999980	.9999977	.9999972	.9999959	.9999901
4.9	.9999990	.9999990	.9999990	.9999990	.9999990	.9999989	.9999988	.9999986	.9999983	.9999975	.9999939
5.0	.9999994	.9999994	.9999994	.9999994	.9999994	.9999993	.9999993	.9999992	.9999990	.9999985	.9999963
5.1	.9999997	.9999997	.9999997	.9999996	.9999996	.9999996	.9999996	.9999995	.9999994	.9999991	.9999978
5.2	.9999998	.9999998	.9999998	.9999998	.9999998	.9999998	.9999998	.9999997	.9999997	.9999995	.9999987
5.3	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999998	.9999998	.9999997	.9999992
5.4	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999999	.9999998	.9999995
5.5	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	1.0000000	.9999999	.9999999	.9999999	.9999997
5.6								1.0000000	1.0000000	.9999999	.9999998
5.7										1.0000000	.9999999
5.8											1.0000000
5.9											
6.0											

$P(K, c)$ = the probability that a point falls inside a circle whose center is at the origin and whose radius is K times the larger standard deviation, c being the ratio of the smaller standard deviation to the larger standard deviation.

Table III

VALUES OF K CORRESPONDING TO CUMULATIVE PROBABILITY P

P \ c	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
.5000	0.67449	0.68199	0.70585	0.74993	0.80785	0.87042	0.93365	0.99621	1.05769	1.11807	1.17741
.7500	1.15035	1.15473	1.16825	1.19246	1.23100	1.28534	1.35143	1.42471	1.50231	1.58271	1.66511
.9000	1.64485	1.64791	1.65731	1.67383	1.69918	1.73708	1.79152	1.86253	1.94761	2.04236	2.14597
.9500	1.95996	1.96253	1.97041	1.98420	2.00514	2.03586	2.08130	2.14598	2.23029	2.33180	2.44775
.9750	2.24140	2.24365	2.25053	2.26255	2.28073	2.30707	2.34581	2.40356	2.48494	2.58999	2.71620
.9900	2.57583	2.57778	2.58377	2.59421	2.60095	2.63257	2.66533	2.71515	2.79069	2.89743	3.03485
.9950	2.80703	2.80883	2.81432	2.83289	2.83830	2.85894	2.88859	2.93347	3.00431	3.11073	3.25525
.9975	3.02334	3.02500	3.03010	3.03898	3.05234	3.07144	3.09871	3.13969	3.20586	3.31099	3.46164
.9990	3.29053	3.29206	3.29673	3.30489	3.31715	3.33464	3.35940	3.39647	3.45698	3.55939	3.71692

Given:

$$\sigma_1 = 15 \text{ meters}$$

$$\sigma_2 = 20 \text{ meters}$$

$$\alpha = 50 \quad .$$

For the computation the following numbers are needed

$$\sigma_1^2 = 225$$

$$\sigma_2^2 = 400$$

$$\sin^2 \alpha = 0.5868 \quad .$$

Substituting in the formula for σ_x^2

$$\sigma_x^2 = \frac{1}{2 \times 0.5868} \left[225 + 400 + \sqrt{625^2 - 4 \times 0.5868 \times 225 \times 400}\right]$$

$$= \frac{1}{1.1736} \left[625 + \sqrt{390,625 - 211,248}\right]$$

$$= 0.85207[625 + \sqrt{179,377}]$$

$$= 0.852[625 + 423]$$

$$= 0.852 \times 1048$$

$$= 893$$

$$\sigma_x = \sqrt{893} = 29.9 \text{ meters}$$

$$\sigma_y^2 = 0.852[625 - 423]$$

$$= 0.852 \times 202$$

$$= 172$$

note that the numbers are the same as for the σ_x^2 calculation except for the minus sign.

$$\sigma_y = \sqrt{172} = 13.1 \text{ meters}$$

$$c = \sigma_y/\sigma_x = 13.1/29.9 = 0.438 \quad .$$

As in the example under Method 1, it is desired to determine the probability of location of this point within a circle of 30 meters radius.

Harter's data is presented in terms of the ratio c, a function K, and probability. The function K, multiplied by the larger of the two standard deviations obtained by this transformation method, gives the value of the radius of the circle of the corresponding value of probability shown in the table. In the example here, the value of the radius of the circle for which a probability is desired is given as 30 meters. Thus we may solve for the proper value of K by the equation:

$$\text{Radius of circle} = K$$

$$30 = K29.9$$

$$K = 1.003 \quad .$$

Then on graph of Fig. 11 (or in Table II) for $K = 1.0$ and $c = 0.44$ (interpolating) read $P = 0.62$ as was obtained from Method 1.

An alternative presentation of Harter's data is given in Fig. 12 and Table III where the parameters are selected so as to provide ready information about the sizes of circles of specific probability value associated with ellipses of varying eccentricities. These are convenient as one often is specifically concerned with the CEP, the 50% probability circle or the 90% circle.

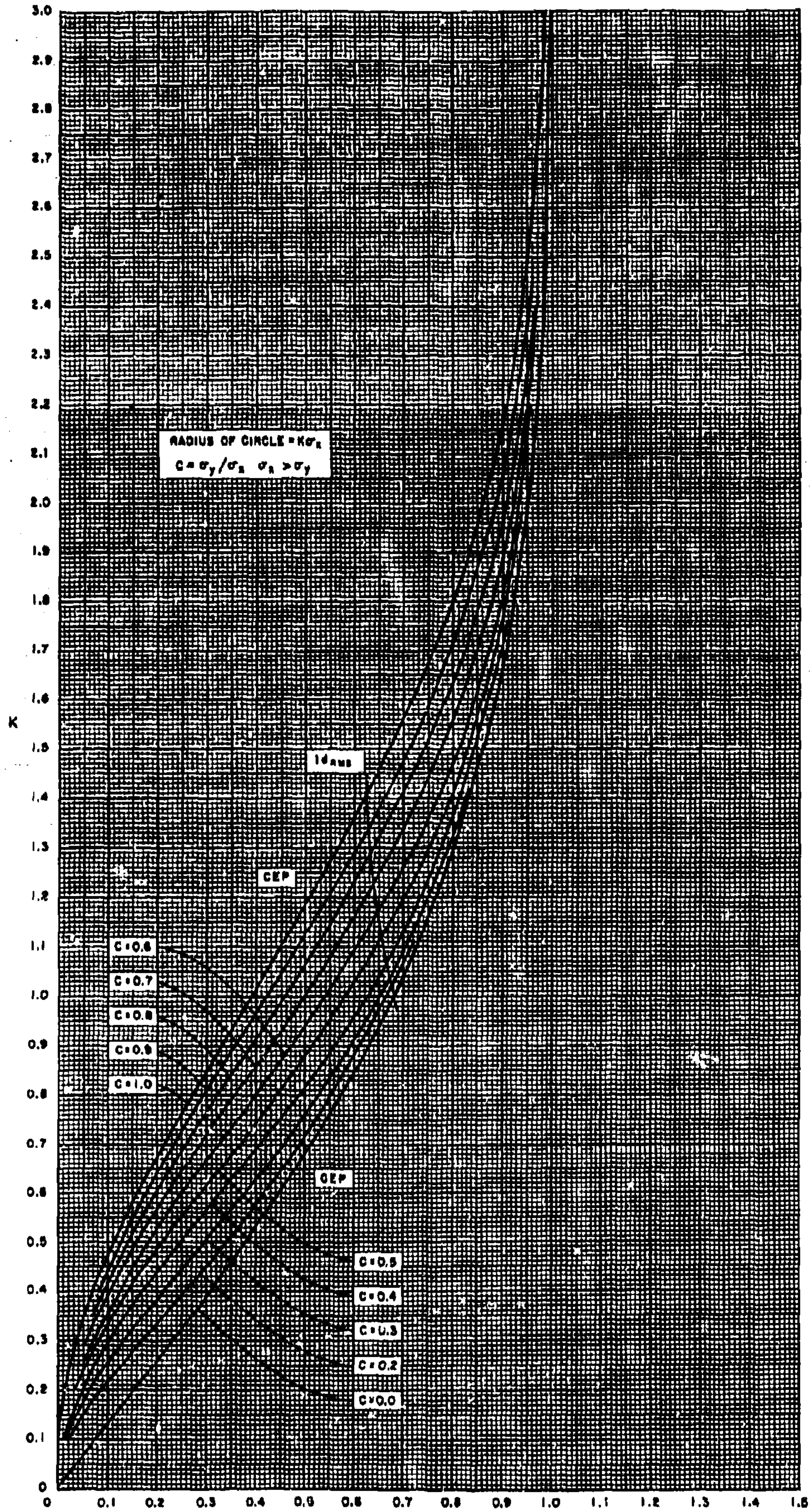

FIG. 11 CIRCULAR ERROR PROBABILITIES FOR ORTHOGONAL ELLIPSES

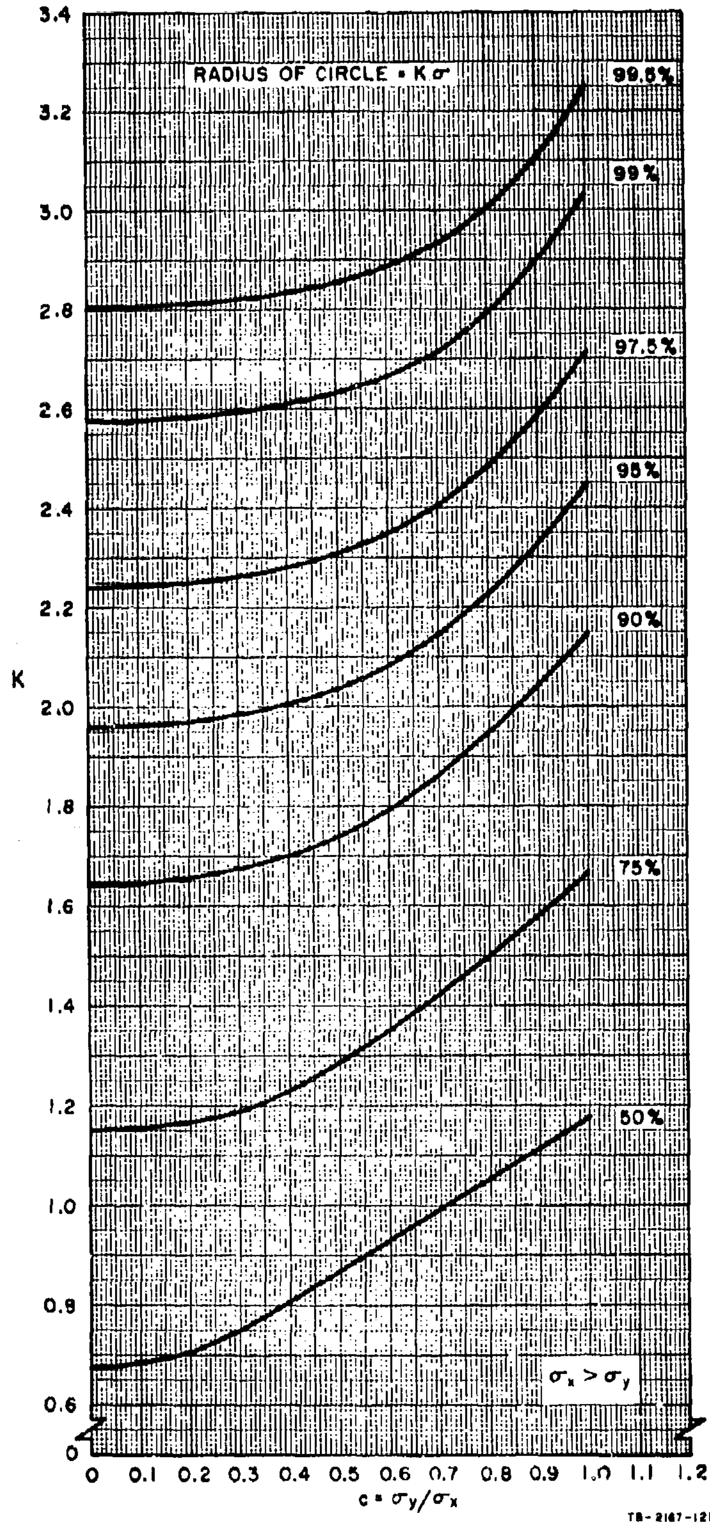

FIG. 12 CONVERSION OF PROBABILITY ELLIPSE TO CIRCLE

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2.3 GEOMETRICAL ERROR CONSIDERATIONS

From the information that can be derived by using these two methods of transformation of elliptical error data, one may develop curves which show for constant values of initial error that the size of a circle of a fixed value of probability varies as a function of the angle of intersection of the lines of position.

To simplify the investigation of geometrical factors, it is initially desirable to consider the special case of $\sigma_1 = \sigma_2 = \sigma$. Under this special condition, the long formulas for σ_x and σ_y may be drastically simplified to facilitate computation as shown below

$$\sigma_x = \frac{\sqrt{2}}{2 \sin \frac{1}{2} \alpha} \sigma \qquad (\sigma_1 = \sigma_2)$$

$$\sigma_y = \frac{\sqrt{2}}{2 \cos \frac{1}{2} \alpha} \sigma \qquad (\sigma_1 = \sigma_2)$$

Taking the ratio of these two values, a simple formula is found for the ratio c.

$$c = \frac{\sigma_y}{\sigma_x} = \tan \frac{1}{2} \alpha \quad .$$

(Detailed derivations for these simplifications may be found in Appendix D.)

Utilizing these simplified formulas, significant parameters of error ellipses have been tabulated in Table IV as a function of the intersection angle α. Using the CEP curve of Fig. 12, values of the CEP have been calculated for each angle, showing that the CEP increases as the angle of intersection decreases. (The tabulation has been carried out only for values of angles less than 90°—the numerical values are symmetrical about this value of angle.) The last column in the table gives the factor by which the CEP for angles less than 90° is greater than the CEP for a right angle. This magnification of error curve is plotted in Fig. 13. A similar computation has been performed for the 90% probability circle as it may be seen that the curve for this value of probability has a slightly

Table IV

SIGNIFICANT PARAMETERS OF ERROR ELLIPSES WHEN $\sigma_1 = \sigma_2 = 1.0$

α	σ_x	σ_y	c	K	CEP	ERROR FACTOR
90	1.0	1.0	1.0	1.177	1.177	1.00
80	1.10	0.924	0.839	1.078	1.186	1.01
70	1.234	0.865	0.700	0.996	1.228	1.042
60	1.414	0.817	0.577	0.914	1.292	1.099
50	1.672	0.782	0.466	0.847	1.420	1.206
45	1.847	0.766	0.414	0.815	1.508	1.281
40	2.06	0.753	0.364	0.783	1.620	1.376
30	2.74	0.733	0.268	0.734	2.01	1.710
20	4.06	0.718	0.176	0.700	2.85	2.42
10	8.11	0.710	0.087	0.680	5.52	4.69

Error Factor = CEP/1.177

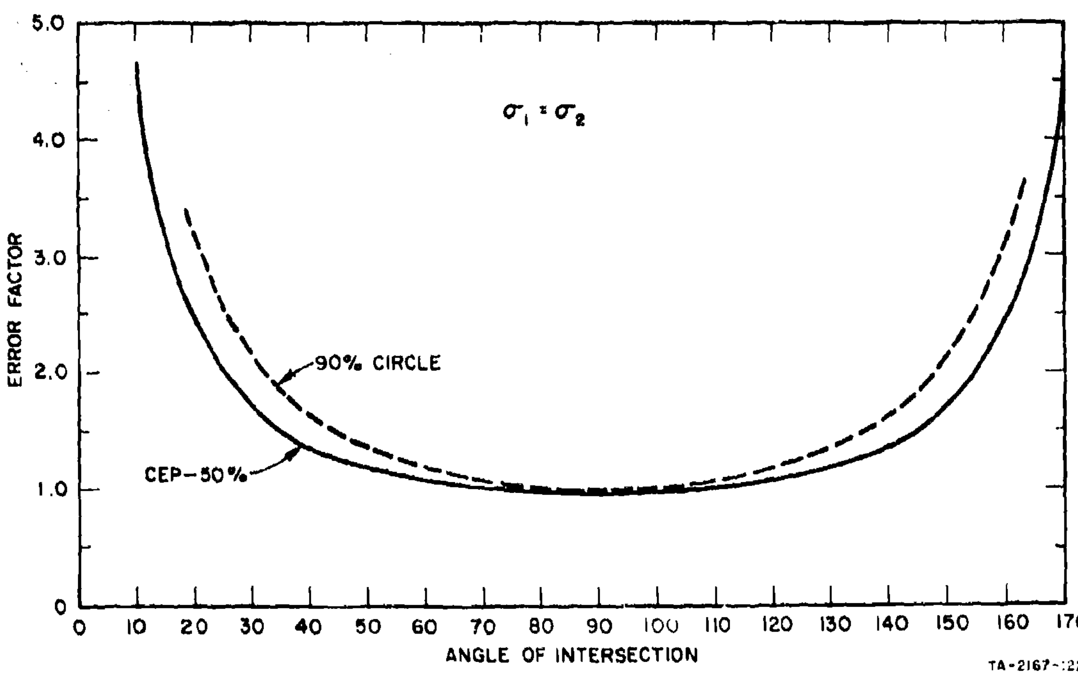

FIG. 13 CEP MAGNIFICATION vs. INTERSECTION ANGLE

Table V

90% CIRCLE ERROR FACTOR

α	c	K	90% R	ERROR FACTOR
90	1.0	2.145	2.145	1.00
80	0.839	1.98	2.18	1.015
70	0.700	1.86	2.30	1.07
60	0.577	1.775	2.51	1.17
50	0.466	1.72	2.88	1.34
45	0.414	1.702	3.15	1.47
40	0.364	1.687	3.47	1.615
30	0.268	1.665	4.53	2.11
20	0.176	1.652	6.72	3.13
10	0.087	1.645	13.35	6.22

Error Factor = 90% R/2.145

differing shape from the CEP curve—see Fig. 12. Values are given in Table V. It is well known in any problem involving position that the best results are obtained when the crossing angle is close to 90°. The curves of Fig. 13 indicate the magnitude of the growth of error as the angle varies from 90°.

It is also of interest to consider an inverse problem—what values of probability result if the radius of the circle is held constant at the minimum value corresponding to that obtaining for the intersection angle = 90°? An answer to this question may be obtained from the probability *vs.* intersection angle curves given under Method 1, Fig. 7.

Along the ordinate $R/\sigma = 1.177$ which corresponds to the CEP for the circular case, one may read the lesser values of probability corresponding to the various intersection angles. Likewise one may also obtain the probability values corresponding to holding a circle the size of the 90% probability circle for the circular case by using the ordinate $R/\sigma = 2.15$ (also equivalent to 1.83 times the CEP). These two curves are plotted in Fig. 14 and the numerical values are given in Table VI. It is to be noted that the probability values are not inversely related to the error factors plotted in the preceding curves. The geometric error factor was shown to be a simple trigonometric function; the probability curves are exponential functions.

Table VI

PROBABILITY DECREASE WITH ANGLE FOR A CIRCLE OF CONSTANT RADIUS $\sigma_1 = \sigma_2$

α	P	P
90	50	90
80	49.4	89.2
70	47.5	86.9
60	44.0	82.4
50	39.5	76
40	37	66
30	25	53
20	17	37
10	8	19

R = CEP/$\underline{90^\circ}$

R = 1.83CEP/$\underline{90^\circ}$

= 90% Probability at 90°

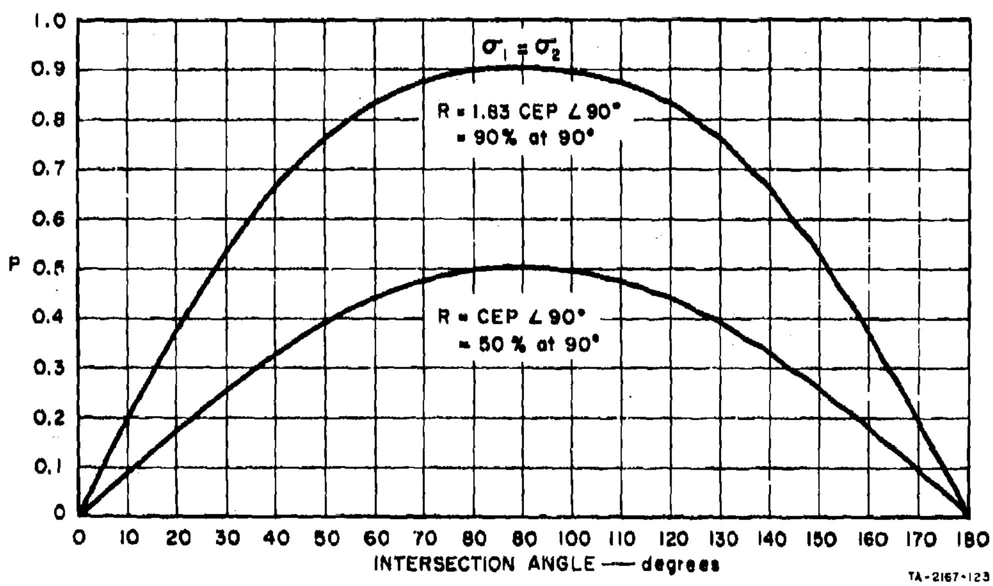

FIG. 14 DECREASE IN PROBABILITY FOR A CIRCLE OF CONSTANT RADIUS vs. ANGLE

3. ANALYSIS OF MULTIPLE ERROR ELLIPSES

The tools developed in the preceding section permit the specification of individual error ellipses about a single point. As shown in Fig. 1 a real problem in position-location involves the consideration of the combination of errors from a number of sources. And as the preceding section showed, in general each of these various sources of error will be expressed as an error ellipse. Following the methods of the previous section, each ellipse can be expressed in terms of the standard deviations along its major and minor axes. The problem of the combination of multiple error ellipses is the determination of the proper method combining a number of individual errors to obtain the total error at some desired point. In the general case, the ellipses will not be oriented relative to one another in any way but a random manner. The exact consideration of this random orientation of the axes of the ellipses complicates the analysis, but it is necessary to obtain accurate answers. More facile, but approximate methods, sometimes seen in the literature, will be discussed at the end of this section. Such approximate methods often result in sizable errors—errors which almost always come out in the wrong direction so that the system appears to be better than it really is.

3.1 SPECIAL CASE—MUTUALLY PARALLEL AXES

Before attempting the analysis of the general case, it is helpful to look at the restricted and unlikely case where all the various ellipses to be considered have their axes mutually parallel. This special case will then lead to the more general case.

Referring again to Fig. 1, there are four error ellipses of interest:

Weapon dispersion

Gun location

Forward observer location

Target location with respect to forward observer.

In a system with these errors we wish to obtain the probability of damage to the target assuming that the shell must land within a circle of 20-meter radius in order to obtain the desired damage level.

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For this initial example it is assumed that all the error ellipses have their axes mutually parallel—aligned north and east, for example—and have the parameters listed in the example shown. Standard deviations for the two axes of each ellipse are listed. The method of obtaining the total error at the target is to obtain the sum of the variances in the two directions and to convert these sums to the standard deviations of the total error ellipse at the target. The desired probability may then be obtained by Method 2 of the preceding section. The calculations are shown in detail.

GIVEN	STANDARD DEVIATIONS		VARIANCES	
	σ_x	σ_y	σ_x^2	σ_y^2
Weapon dispersion	3 m	40 m	9	1600
Gun location	10	15	100	225
Forward observer	15	20	225	400
Target location	30	10	900	100
	Sums of variances		1234	2325

Take square roots of variances to obtain new standard deviations of total error ellipse at target.

$$\sigma_x = \sqrt{1234} = 35.1 \text{ m}$$
$$\sigma_y = \sqrt{2325} = 48.2 \text{ m}$$

From Method 2, Section 2:

$$c = 35.1/48.2 = 0.729$$
$$\text{Radius of circle} = K\sigma_{\text{larger}}$$
$$20 = 48.2\ K$$
$$K = 20/48.2 = 0.415$$

Then from the graph of Fig. 11 for $K = 0.415$ and $c = 0.73$ (interpolating) read $P = 0.11$.

In this case of the combination of several ellipses σ_y has turned out to be larger than σ_x. In such cases the factor K is always to be multiplied by the larger of the two values of sigma, σ_x or σ_y to obtain the radius of the probability circle. The formulas given on Fig. 12 for simplicity are stated in terms of σ_x always being the larger. The

formulas assure this condition for any single ellipse taken alone without reference to other ellipses. However, when ellipses are combined, either standard deviation may turn out to be the larger.

By way of comparison it is interesting to calculate how much of the error is contributed by location measurement errors and how much is contributed by the dispersion of the weapon. If we consider the weapon dispersion ellipse alone

$$c = 3/40 = 0.075$$

$$\text{Radius of circle} = K\sigma_{\text{larger}}$$

$$20 = 40\ K$$

$$K = 0.5$$

Then on the graph of Fig. 11 for $K = 0.5$ and $c = 0.075$ (interpolating) read $P = 0.37$.

Thus with perfect location of all elements, this gun with the stated dispersion (105 mm, mid-range) has a 37% probability of landing a shell within a circle of 20-meter radius. But, when the three location errors are combined with the dispersion, the probability falls to 11%.

The numbers shown are realistic for a system of good accuracy and better than most performance of today.

This method of adding variances along the two axes at right angles, down range and cross range, is the standard method of preparing an error budget for a weapon system. The method, however, is not sufficient when one wishes to combine ellipses having random orientations of their axes. And since statistical distributions are involved, simple trigonometric resolutions from one set of axes to another are insufficient.

3.2 GENERAL CASE—RANDOM ORIENTATION OF AXES

In the general case of random orientation of axes in any number of error ellipses, a more complex procedure for combination is necessary than that described in the previous subsection. Briefly, a reference set of axes must be chosen and the orientation of each error ellipse with respect to these axes must be determined. Then the variances along these axes must be computed, a procedure which will also involve cross product terms (See Appendix B). The special variances and the cross

product functions may then be added to obtain the corresponding functions of the final ellipse. From these two variances and the cross product function of the final ellipse, which are associated with the arbitrarily chosen set of axes, final values of σ_x and σ_y along the major and minor axes of the ellipse may be calculated. Necessary formulas are given below. Illustrative examples will clarify the description. The labor involved is considerable for any real example. For this reason, a computer program was developed to permit such calculations to be run off in large quantities as a part of the systems evaluation program. Derivations of the formulas presented are to be found in Appendix D.

An example is illustrated in Fig. 15. The three smaller ellipses are the given inputs to the problem and the large ellipse on the right represents the combination of the three smaller ones. Each of the three small ellipses is described in terms of its own σ_x and σ_y. Also shown are the angles between the x-axis of each ellipse and the arbitrarily selected reference axes which are designated the w and z axes. To obtain the parameters of the final ellipse, variances for each ellipse along the w and z axes are calculated. Because the axes are not those of the individual ellipses, an additional function ρ, the cross product function

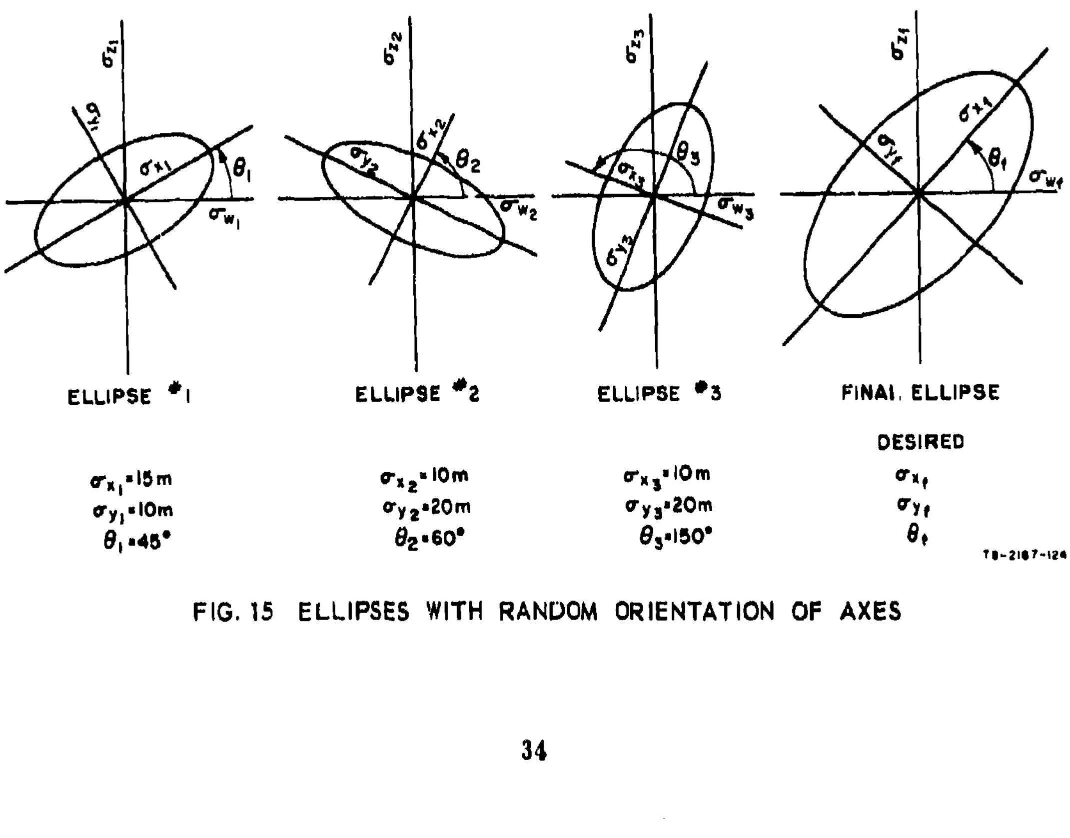

FIG. 15 ELLIPSES WITH RANDOM ORIENTATION OF AXES

is also required for each ellipse. These three functions, the two variances and the cross product function, from each ellipse are then added to obtain the corresponding functions in the final ellipse. Functions for the final ellipse are designated with the second subscript f. These three functions in the final ellipse may then be converted to the σ_{xf} and σ_{yf} of that ellipse along its major and minor axes. The calculations are tabulated for this illustrative example on the next few pages, following the listing of the necessary formulas. The problem illustration gives all intermediate values needed but does not show every individual calculation required. The extensive amount of work necessary for such a simple problem clearly shows the need for a mechanized computer solution. The simple problem illustrated requires an hour or more of hand labor with tables and slide rule—the computer handles it in three seconds.

The necessary formulas for this calculation are given below without derivations. Further discussion and derivations will be found in Appendix E. The formulas are given in general notation using the letter i to represent the general ellipse. In the use of the formulas, $i = 1,2,3,\ldots n$ according to the number of ellipses involved. The formulas are most conveniently expressed in terms of some auxiliary functions which are computed first.

Define

$$A_i = \frac{\cos^2 \theta_i}{\sigma^2_{xi}} + \frac{\sin^2 \theta_i}{\sigma^2_{yi}}$$

$$B_i = \frac{\cos^2 \theta_i}{\sigma^2_{yi}} + \frac{\sin^2 \theta_i}{\sigma^2_{xi}}$$

$$C_i = \sin \theta_i \cos \theta_i \left(\frac{1}{\sigma^2_{yi}} - \frac{1}{\sigma^2_{xi}} \right)$$

Then

$$\rho_i = \frac{C_i}{\sqrt{A_i B_i}}$$

$$\sigma^2_{wi} = \frac{1}{(1 - \rho_i^2) A_i}$$

$$\sigma^2_{zi} = \frac{1}{(1 - \rho_i^2) B_i}$$

These last three formulas give the variances and the cross product function for each ellipse in terms of the w and z axes. These are then combined according to the next three formulas to obtain the corresponding functions for the final ellipse.

$$\sigma^2_{wf} = \sum_{i=1}^{n} \sigma^2_{wi}$$

$$\sigma^2_{zf} = \sum_{i=1}^{n} \sigma^2_{zi}$$

$$\rho_f = \frac{1}{\sigma_{wf}\sigma_{zf}} \sum_{i=1}^{n} (\rho_i \sigma_{wi} \sigma_{zi})$$

We now have the parameters of the final ellipse in terms of the w and z axes. To eliminate the cross product function ρ_f and obtain σ_{xf} σ_{yf} along the major and minor axes of the final ellipse, we use the formulas below.

$$\sigma^2_{xf} = \frac{1}{2}\left[\sigma^2_{wf} + \sigma^2_{zf} + \sqrt{(\sigma^2_{wf} + \sigma^2_{zf})^2 - 4\sigma^2_{wf}\sigma^2_{zf}(1 - \rho^2_f)}\right]$$

$$\sigma^2_{yf} = \frac{1}{2}\left[\sigma^2_{wf} + \sigma^2_{zf} - \sqrt{(\sigma^2_{wf} + \sigma^2_{zf})^2 - 4\sigma^2_{wf}\sigma^2_{zf}(1 - \rho^2_f)}\right]$$

The orientation of the final ellipse with respect to the w and z axes is given by the formula below.

$$\tan 2\theta_f = \frac{2\,\rho_f \sigma_{wf} \sigma_{zf}}{\sigma^2_{wf} - \sigma^2_{zf}}$$

The numerical example follows.

GIVEN CONDITIONS	ELLIPSE #1	ELLIPSE #2	ELLIPSE #3
σ_{xi}	15 m	10 m	10 m
σ_{yi}	10 m	20 m	20 m
θ_i	45°	60°	150°
CALCULATED			
σ^2_{xi}	225	100	100
σ^2_{yi}	100	400	400

$\cos^2\theta_i$	0.5000	0.2500	0.7500
$\sin^2\theta_i$	0.5000	0.7500	-0.2500
$\sin\theta_i \cos\theta_i$	0.5000	0.4330	-0.4330
$\dfrac{\cos^2\theta_i}{\sigma_{xi}^2}$	0.0002222	0.0002500	0.007500
$\dfrac{\sin^2\theta_i}{\sigma_{yi}^2}$	0.005000	0.001875	0.0006250
A_i	0.007222	0.004375	0.008125
$\dfrac{\cos^2\theta_i}{\sigma_{yi}^2}$	0.005000	0.0006250	0.001875
$\dfrac{\sin^2\theta_i}{\sigma_{xi}^2}$	0.002222	0.007500	0.002500
B_i	0.007222	0.008125	0.004375
$A_i B_i$	5.216×10^{-5}	3.555×10^{-5}	3.555×10^{-5}
$\sqrt{A_i B_i}$	0.007222	0.005962	0.005962
$\dfrac{1}{\sigma_{yi}^2}$	0.1000	0.002500	0.002500
$\dfrac{1}{\sigma_{xi}^2}$	0.004444	0.01000	0.01000
$\dfrac{1}{\sigma_{yi}^2} - \dfrac{1}{\sigma_{xi}^2}$	0.005556	-0.007500	-0.007500
C_i	0.002778	-0.003750	0.003750
ρ_i	0.3849	-0.5451	0.5451
ρ_i^2	0.1481	0.2971	0.2971
$1 - \rho_i^2$	0.8519	0.7029	0.7029
σ_{wi}^2	162.6	325.2	175.1
σ_{zi}^2	162.6	175.1	325.2
σ_{wi}	12.75 m	18.03 m	13.23 m
σ_{zi}	12.75 m	13.23 m	18.03 m

$$\sigma^2_{wf} = 162.6$$
$$325.2$$
$$\underline{175.1}$$
$$662.9$$

$$\sigma^2_{sf} = 162.6$$
$$175.1$$
$$\underline{325.2}$$
$$662.9$$

$$\sigma_{wf} = 25.74 \text{ m} \qquad \sigma_{sf} = 25.74 \text{ m}$$

$$\rho_f = \frac{1}{25.74 \times 25.74} [0.3849 \times 12.76 \times 12.76 - 0.5451 \times 18.03 \times 13.23 + 0.541 \times 13.23 \times 18.03]$$

$$= \frac{1}{662.9} \times 0.3849 \times 162.6$$

$$= \frac{62.58}{662.9} = 0.09441$$

$$\rho_f^2 = 0.0089132$$

$$(1 - \rho_f^2) = 0.9910968$$

Substituting these values of σ_{wf}, σ_{sf}, and ρ_f into the formula for σ_{xf}

$$\sigma^2_{xf} = \frac{1}{2} \left[662.9 + 662.9 + \sqrt{(662.9 + 662.9)^2 - 4 \times 662.9^2 (9.99109)}\right]$$

$$= \frac{1}{2} \left[1325.8 + \sqrt{4 \times 662.9^2 \times 0.008913}\right]$$

$$= \frac{1}{2} [1325.8 + 125.2]$$

$$= \frac{1}{2} \times 1451$$

$$= 725.5$$

$$\sigma_{xf} = \sqrt{725.5} = 26.93 \text{ m}$$

$$\sigma^2_{yf} = \frac{1}{2} [1325.8 - 125.2]$$

$$= \frac{1}{2} \times 1200.6 = 600.3$$

$$\sigma_{yf} = \sqrt{600.3} = 24.5 \text{ m}$$

$$\tan 2\,\theta_f = \frac{2 \times 0.09441 \times 25.74 \times 25.74}{662.9 - 662.9}$$

$$= \frac{\text{numerator}}{0}$$

$$= \infty$$

$$\text{arctan } \infty = 90^\circ$$

$$2\,\theta_f = 90^\circ$$

$$\theta_f = 45^\circ$$

The computer solution of this same problem gives answers differing only by one in the fourth place, thus confirming the results.

Because of the labor involved in the foregoing calculations, some references have indicated that errors may be combined by converting each error distribution to the CEP at that point according to Method 2 and then combining the individual CEP values root-sum-square. This method almost always has been found to give too small a value of the CEP at the final ellipse. Conversion to individual circles eliminates the effect of orientation of the ellipse which is an important consideration in combination. To illustrate, the same three ellipses used in the previously lengthy example will be handled by this simplified method and compared with the answer obtained by converting the final ellipse to a CEP.

	ELLIPSE #1	ELLIPSE #2	ELLIPSE #3
σ_{xi}	15	10	10
σ_{yi}	10	20	20
C	0.667	0.50	0.50
K	0.956	0.870	0.870
CEP	14.35	17.4	17.4
CEP^2	206	302	302

Sum of $(CEP's)^2$

$$\begin{array}{r} 206 \\ 302 \\ \underline{302} \\ 810 \end{array}$$

CEP of final location $= \sqrt{810} = 28.45$ m

Now compare this with CEP of final ellipse as just computed

$$\sigma_{xf} = 26.93$$

$$\sigma_{yf} = 24.5$$

$$C = 24.5/26.93 = 0.948$$

$$K = 1.144$$

$$\text{CEP} = 1.144 \times 26.93 = 30.8 \text{ m}$$

The error in this case is relatively small while the simplified method gives an answer that is 8% too small. Other sample calculations indicate that errors of up to 20% can occur using this simplified method. Approximate calculations of the CEP are discussed in more detail in Appendix A.

APPENDIX A

MEASURES OF ERROR

APPENDIX A

MEASURES OF ERROR

1. INTRODUCTION

This appendix describes the various terms used as measures of error in more detail than the discussion in the body of this Memorandum. In addition, other terms often noted in the pertinent literature are defined and related to the terms used in this report.

The reader is cautioned to read analyses of systems accuracy with care. The literature examined by the SRI project team in the progress of this study sometimes contains numerical errors. More serious, however, incomplete or misleading definitions of terms are often found, and, occasionally, incorrect definitions have been noted. In other references it may be impossible to tell exactly what was meant by a particular measure of error. The discussion of measures of error in this appendix is intended to help clear away such misunderstanding and confusion.

2. ONE DIMENSIONAL ERROR TERMS

Although the basic problem of position location is concerned with the two dimensions necessary to describe an area, one dimensional error measures are commonly applied to each of the two dimensions involved. In fact, as shown in the discussion in the body of this memorandum, it is most convenient to do this to permit a truly general approach to the consideration of error ellipses. Thus the measures of error concerned with one dimensional of Gaussian distributions are important. The following terms are frequently met and each is described in following paragraphs: standard deviation, RMS error, sigma, probable error, and variance.

2.1 Standard Deviation, RMS Error, Sigma (σ)

The three terms all mean the same. The basic equation of the normal distribution indicates the use of the Greek letter sigma from which the shorthand use of sigma for standard deviation arises.

$$f(x) = \frac{1}{2\pi\sigma} e^{-\frac{1}{2\sigma^2}(x-a)^2}$$

Standard deviation of a measurement system is a property that may be determined experimentally. If a large number of measurements of the same quantity—a length, for example—are made and compared with a standard, the standard deviation is the square root of the sum of the squares of the deviations from the mean or average value divided by the number of measurements taken. Symbolically, this operation is represented as

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n} (x_u - a)^2}{n-1}}$$

The term rms (root-mean-square) error comes from this latter method of computation.

Numerically, one sigma corresponds to 68% of the distribution: that is, if a large number of measurements were made of a given quantity, 68% of the errors would be no greater than the value of one standard deviation. Likewise 2σ corresponds to 95% of the total errors, and 3σ to 99.6% of the total errors.

2.2 Probable Error

This term is identical in concept to standard deviation when considered as the rms error determined after a series of measurements. The term differs from standard deviation in that it refers to that value corresponding to the median error; no more than half the errors in the measurement sample are greater than the value of the probable error. Linear probable error is related to standard deviation by a multiplying factor. One probable error equals 0.6745 times one standard deviation. Probable error is used in Army artillery manuals as a measure of weapon component errors, such as the range and deflection errors associated with a particular weapon and ammunition. Industrial practice in the United States also employs the probable error when a measurement is re-reported in the manner of 173.23 ± 0.05 ft. The 0.05 ft is to be

interpreted as a probable error of the measurement. Probable error has not been used in the analyses of this study for it is not as convenient an error measure to handle mathematically as the standard deviation.

2.3 VARIANCE

This term is met most frequently in detailed mathematical discussions. The term refers to the square of a standard deviation. It is useful in simplifying the algebra of some complex mathematical derivations (see Appendix B for examples). It also is a convenient concept when preparing an error budget made up of many separate components of error for the individual variances may be added directly to obtain the total variance.

3. TWO-DIMENSIONAL ERROR TERMS

Terms similar or identical in words to those used for one-dimensional error descriptions are also used with two-dimensional or bivariate error descriptions. However, in the two-dimensional case, not all of these terms have the same meaning as before and considerable care is needed to avoid confusion.

3.1 STANDARD DEVIATION OR SIGMA

These two terms, used interchangeably, have a definable meaning only in the specific case of the circular normal distribution where $\sigma_x = \sigma_y$

$$P_R = 1 - e^{-\frac{R^2}{2\sigma^2}}$$

In the case of the circular normal distribution, the standard deviation σ is equivalent to the standard deviation along both orthogonal axes. Because we are here concerned with a radial distribution, the total distribution of errors involves different numbers from those of the linear case. In the circular case, 1 σ error indicates that 39.3% of the errors would not exceed the value of the 1 σ error; 86.4% would not exceed the 2σ error, and 98.8% would not exceed the 3 σ error.

Because the usual case where there are two-dimensional distributions is that the standard deviations along the two axes are different, resulting

in an elliptical distribution, the circular standard deviation is less useful than the linear standard deviation. It is more common to describe two-dimensional distributions by the two separate one-dimensional standard deviations associated with each error axis. References often do not make this distinction, however, referring to the position accuracy of a system as "600 ft (2σ)," for example. Such a description leaves the reader wondering whether the measure is circular error, in which case the numbers describe the 86% probability circle, or whether the numbers are to be interpreted as one-dimensional sigmas along each axis, in which case the 95% probability circle is indicated (assuming the distribution to be circular, which actually it may not be). The analyses of this report have, in general, used the two separate linear standard deviations as error measures. Where specific circular measures have been used, they are so noted carefully to avoid confusion. (See next subsection.)

The term RMS error when applied to two-dimensional errors, does not have the same meaning as the standard deviation. The term is often used in the literature, although it has an ambiguous meaning in relation to terms of probability. For this reason its use has been deprecated in this study. The term is discussed separately in a following subsection.

3.2 CEP (Circular Error Probable—Also Sometimes CPE, Circular Probable Error)

In a circular normal distribution this term refers to the radius of a circle containing 50% of the sample of the individual measurements being made, or the radius of the circle inside of which there is a 50% probability of being located. This is a common measure often used with weapon systems and position location systems.

The term CEP is also used to indicate the radius of a circle inside of which there is a 50% chance of being located, even though the actual error figure is an ellipse (Fig. A-1). The body of the Memorandum describes the method of obtaining such CEP equivalents when given ellipses of varying eccentricities. Curves and tables are furnished to perform this calculation. In the literature, despite the availability of these curves and tables, approximations are often made for this calculation of a CEP when the actual error distribution is elliptical. Several of these approximations are indicated and plotted for comparison with the exact curve in Fig. A-2. Of the various approximations shown, the top curve, the one which diverges the most rapidly, appears to be the most commonly

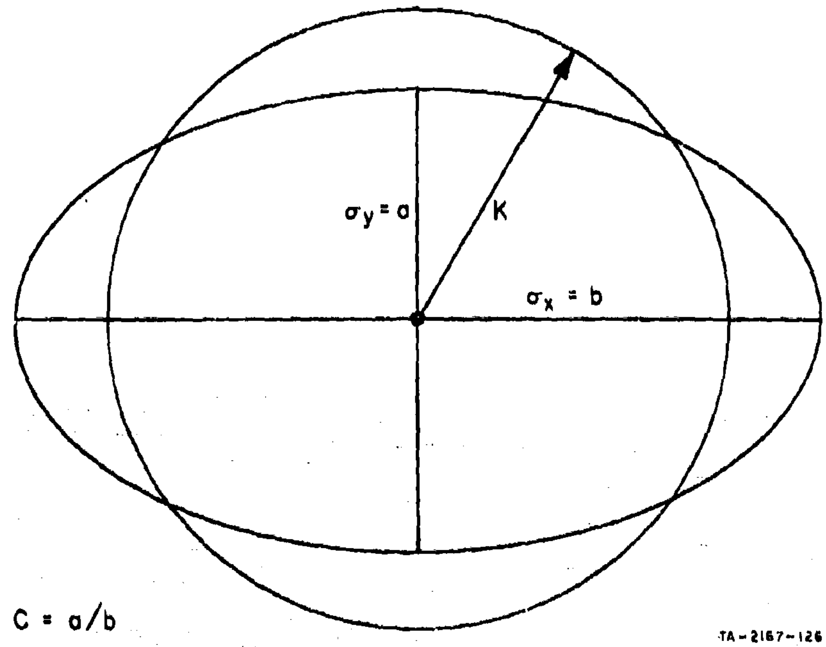

FIG. A-1 ERROR ELLIPSE AND CIRCLE OF EQUIVALENT PROBABILITY

used in published systems analyses. Use of the curves and tables given in the body of this Memorandum is recommended to avoid such problems of approximations.

Another factor of interest concerning the relationship of the CEP to various ellipses is that the area of the CEP circle is always greater than the basic ellipse. Calculations made using the values of the tables given in the body of the Memorandum are given in Table A-1 where it may readily be seen that the divergence between the actual area of the ellipse of interest and the circle of equivalent probability increases as the ellipse becomes thinner and more elongated. This fact provides a powerful reason for the method of analysis used in this study of considering the ellipses directly, especially when a number of ellipses are involved in the determination of a final probability figure.

Table A-1

COMPARISON OF AREAS OF 50% ELLIPSES OF VARYING ECCENTRICITIES WITH AREAS OF CIRCLES OF EQUIVALENT PROBABILITY

$C = a/b$	AREA OF 50% ELLIPSE	AREA OF EQUIVALENT CIRCLE
0.0	0	1.43
0.1	0.437	1.46
0.2	0.874	1.56
0.3	1.31	1.76
0.4	1.75	2.06
0.5	2.08	2.37
0.6	2.62	2.74
0.7	3.06	3.12
0.8	3.49	3.52
0.9	3.93	3.94
1.0	4.37	4.37

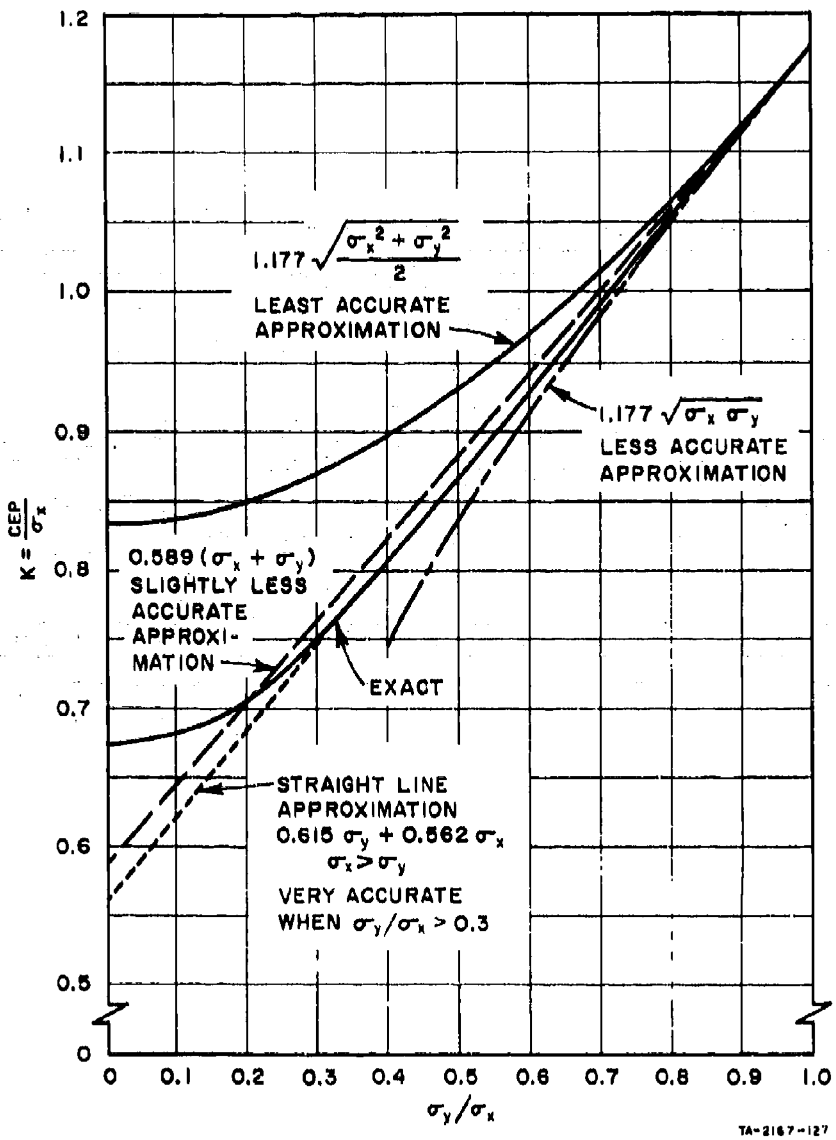

FIG. A-2 CEP FOR ELLIPTICAL ERROR DISTRIBUTION APPROXIMATIONS

The effect of conversion of ellipses to the equivalent CEP values is well illustrated in the combination of ellipses of different orientations. If two equal ellipses whose relative orientation is 90° are combined, the result is a circle. As mentioned in the last section of the body of the Memorandum, combination of ellipses is sometimes calculated by converting each individually to its equivalent CEP, and then combining the individual CEP values root-sum-square. The simple example shown here indicates that a considerable degree of error can result from such a combination of separately obtained CEP values when compared with the CEP obtained from the combined figure.

ELLIPSE #1	ELLIPSE #2
$\sigma_{x1} = 1$	$\sigma_{x2} = 10$
$\sigma_{y1} = 10$	$\sigma_{y1} = 1$
$C = 0.1$	$C = 0.1$
$K = 0.681$	$K = 0.681$
$\text{CEP} = K\sigma_{y1}$	$\text{CEP} = K\sigma_{x2}$

In each case CEP = 6.81. If the two values of CEP are combined root-sum-square the result is $6.81\sqrt{2} = 9.65$. If we now combine variances to obtain the combination, the results are:

$$\sigma^2_{x1} = 1 \qquad \sigma^2_{y1} = 100$$

$$\sigma^2_{x2} = 100 \qquad \sigma^2_{y2} = 1$$

$$\sigma^2_{xf} = 101 \qquad \sigma^2_{yf} = 101$$

$$\sigma_{xf} = 10.04 \qquad \sigma_{yf} = 10.04$$

Thus, the final figure is a circle.

$$C = 1.0$$

$$K = 1.177$$

$$\text{CEP} = 1.177 \times 10.04 = 11.9.$$

This figure is the correct one and the answer obtained by combination of individual CEP values is $1 - (9.65/11.9) = 19\%$ too low.

The value of the CEP may be related to the radius of other values of probability circles analytically for the case of the circular normal distribution by solving the basic equation for various values of probability. For this special case of the circular normal distribution these relationships are shown drawn to scale in Fig. A-3 with the associated values tabulated in Table A-2.

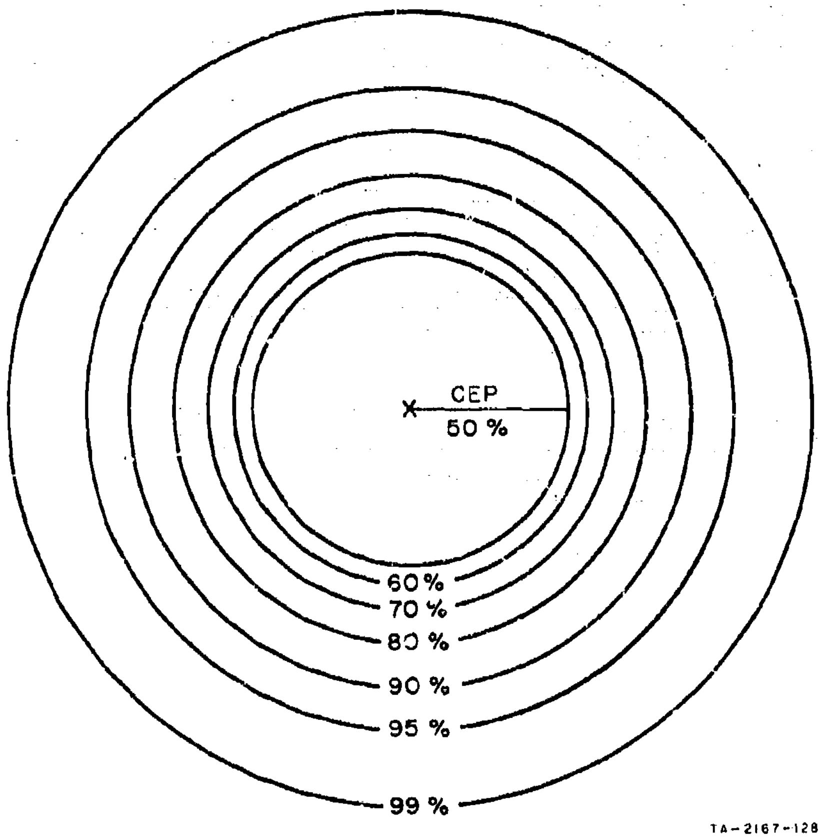

FIG. A-3 RELATIONSHIP BETWEEN CEP AND OTHER PROBABILITY CIRCLES

The derivation of these values is shown in the following analysis. First, the factor relating the CEP to the circular sigma is derived, then, as a second example, the relationship between the 75% probability circle and the circular sigma. The ratio between these two values is then the value shown in Table A-2 for the 75% value.

Table A-2

RELATIONSHIP BETWEEN CEP AND RADII OF OTHER PROBABILITY CIRCLES OF THE CIRCULAR NORMAL DISTRIBUTION

MULTIPLY VALUE OF CEP BY	TO OBTAIN RADIUS OF CIRCLE OF PROBABILITY
1.150	60%
1.318	70%
1.414	75%
1.524	80%
1.655	85%
1.823	90%
2.079	95%
2.578	99%

Circular normal distribution equation is

$$P(R) = 1 - e^{-\frac{R^2}{2\sigma^2}}$$

and

$$\text{CEP} \equiv P(R) = 0.5$$

$$1 - e^{-\frac{R^2}{2\sigma^2}} = 0.5$$

$$e^{-\frac{R^2}{2\sigma^2}} = 0.5$$

Take natural logarithm of both sides

$$ln\left(e^{-\frac{R^2}{2\sigma^2}}\right) = ln\ 0.5$$

$$-\frac{R^2}{2\sigma^2} = ln\ 0.5$$

$$\frac{R^2}{2\sigma^2} = ln\ 2 \qquad (ln\ 0.5 = -\ ln\ 2)$$

$$R = \sigma\sqrt{2\ ln\ 2} = \sigma\sqrt{1.386}$$

$$R = 1.177\sigma$$

$$ln = \log e = \text{natural logarithm}$$

$$ln\ 2 = 0.6931.$$

For the 75% probability circle

$$1 - e^{-\frac{R^2}{2\sigma^2}} = 0.75$$

$$e^{-\frac{R^2}{2\sigma^2}} = 0.25$$

$$\ln\left(e^{-\frac{R^2}{2\sigma^2}}\right) = \ln 0.25$$

$$\frac{R^2}{2\sigma^2} = \ln 4$$

$$R = \sigma\sqrt{2 \ln 4}$$

$$R = 1.665\sigma$$

$$\frac{R75}{R50} = \frac{1.665\sigma}{1.177\sigma} = 1.414$$

The factors tabulated in Table A-2 are sometimes used in the literature to relate varying probability circles when the basic distribution is not circular, but elliptical. That such a procedure is inaccurate may be seen by the curves of Fig. A-4. These curves were prepared from the values of Table III in the body of the Memorandum. It may be seen that the errors involved are small when small ellipticities are involved. But the errors increase significantly when both high values of probability are desired and when the ellipticity increases in the direction of long, narrow distributions.

3.3 Correlation Coefficient

In many statistical references the presence of a cross-product term is indicated as a correlation between the two variables, such as z_1 and z_2. As used in this statistical sense the term *correlation* does not imply that the variables z_1 and z_2 are connected in any physical way. The two variables are independent and a change in one does not affect the other. The factor ρ (rho) which appears in equations relating multiple ellipses to an arbitrary set of axes is that factor often called the *correlation coefficient* in statistical literature. Because *correlation* can be

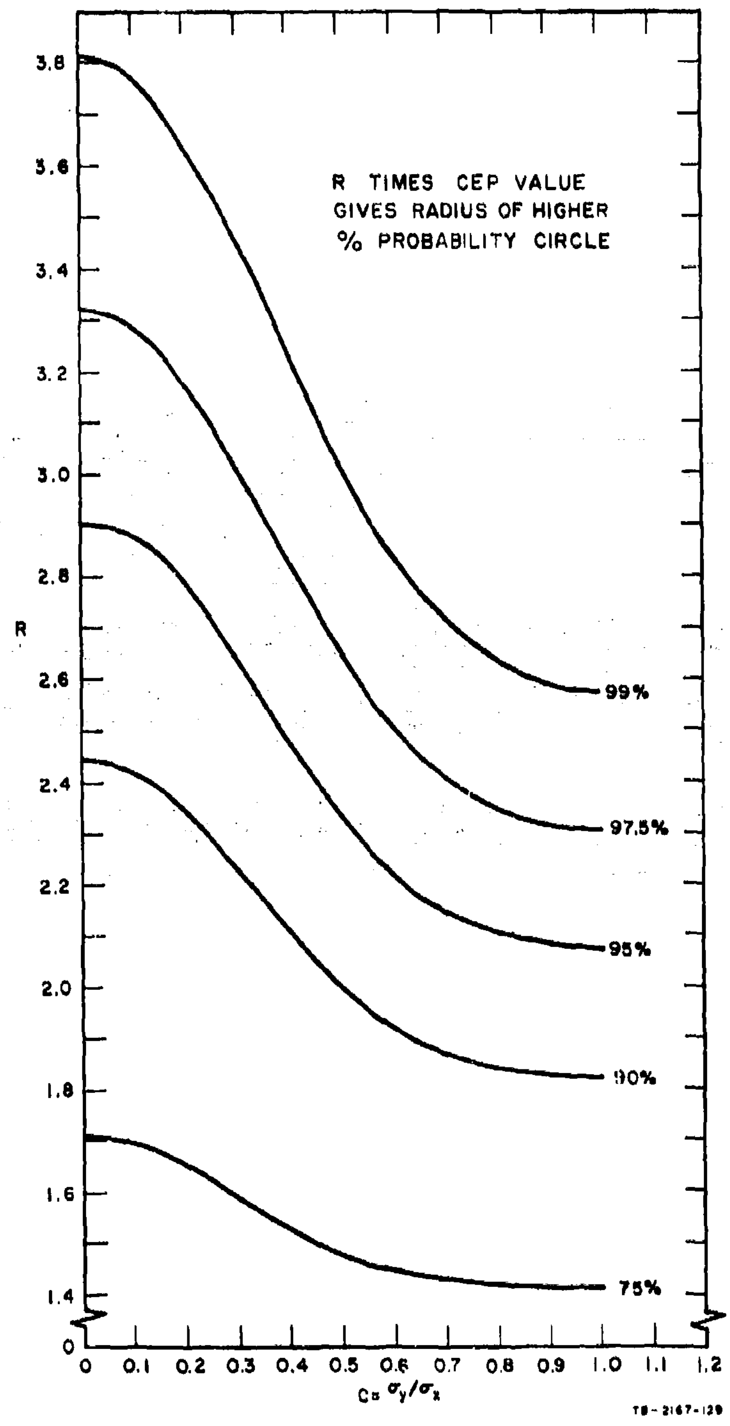

FIG. A-4 RELATION OF PROBABILITY CIRCLES TO CEP vs. ELLIPTICITY

misinterpreted as implying a physical interrelationship which does not actually exist, the use of the term has been avoided in this study, despite the consequent need for the circumlocution of "cross-product term in $z_1 z_2$."

The term *correlation* is also used in some of the references to indicate the case when there *is* an interrelationship between the two variables, a condition that obtains in some physical systems where two signals may be synchronized to a third. Hence, one must be very careful when using the term to insure a proper understanding of the mathematics. Because the term correlation is thus subject to possible confusion and contradiction, its use when referring to a cross-product term has been avoided in this study. When it occurs, it will refer solely to an actual physical interrelationship.

3.4 Radial or RMS Error, d_{rms}

The terms radial error, RMS error, and d_{rms} are identical in meaning when applied to two-dimensional errors. Figure A-5 illustrates the definition of d_{rms}. It is seen to be the square root of the sum of the squares of the one sigma error components along the major and minor axes of a probability ellipse. The figure details the definition of 1 d_{rms}. Similarly, other values of d_{rms} may be derived by using the corresponding values of sigma. The measure d_{rms} is *not* equal to the square root of the sum of the squares of the σ_1 and σ_2 that are the basic errors connected with the lines of position of a particular position location system. The procedures described in the main portion of this Memorandum and derived in Appendix D must first be utilized to obtain the values shown as σ_x and σ_y in Fig. A-5. The three terms used as a measure of error, RMS error, radial error, and d_{rms} are somewhat confusing because they do not correspond to a fixed value of probability for a given value of the error measure. The terms can be conveniently related to other error measures only when $\sigma_x = \sigma_y$ and the probability figure is a circle. In the more common elliptical cases, the probability associated with a fixed value of d_{rms} varies as a function of the eccentricity of the ellipse. 1 d_{rms} is defined as the radius of the circle obtained when $\sigma_x = 1$ in Fig. A-5 and σ_y varies from zero to one. Likewise, 2 d_{rms} is the radius of the circle obtained when $\sigma_x = 2$ and σ_y varies from zero to two. Values of the length of the radius d_{rms} may be calculated as shown in Table A-3. From these values the associated probabilities may

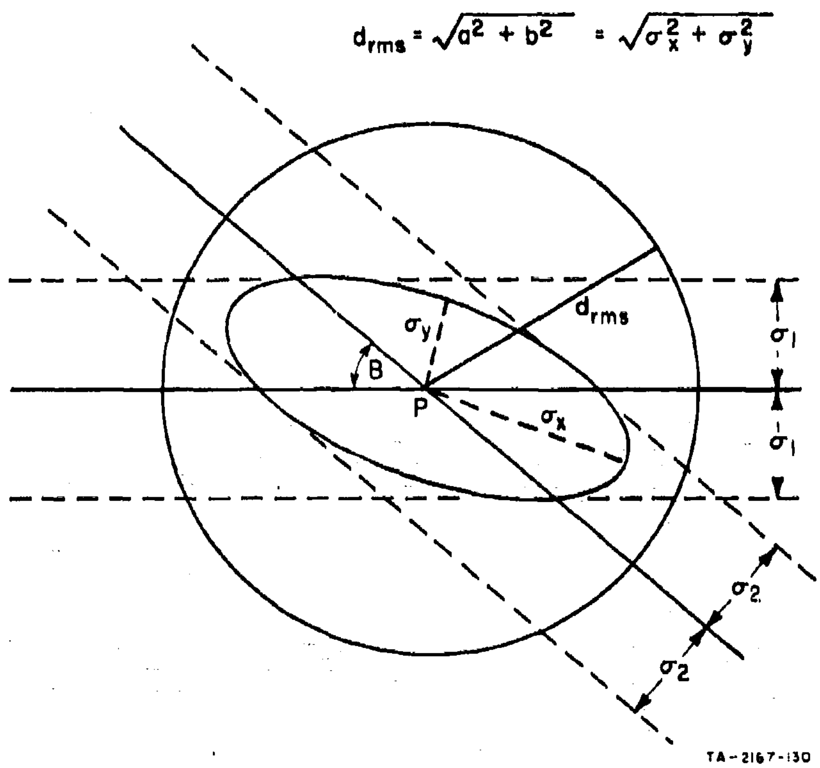

FIG. A-5 ILLUSTRATION OF ROOT MEAN SQUARE ERROR

be determined from the tables of the body of this annex. The variations of probability associated with the values 1 d_{rms} and 2 d_{rms} are shown in the curves of Figs. A-6 and A-7. Fig. A-8 shows the lack of a constant relationship in a slightly different way. Here the ratio d_{rms}/CEP is plotted against the same measure of ellipticity. The three figures show graphically that there is not a constant value of probability associated with a single value of d_{rms}. While this variation is not great, it is felt to be unnecessarily confusing. Thus the measure d_{rms} has not been used in this study.

Table A-3

d_{rms} CALCULATIONS

σ_y	σ_x	LENGTH OF 1 d_{rms}	PROBABILITY 1 d_{rms}	PROBABILITY 2 d_{rms}
0.0	1.0	1.000	0.683	0.954
0.1	1.0	1.005	0.682	0.955
0.2	1.0	1.020	0.682	0.957
0.3	1.0	1.042	0.676	0.961
0.4	1.0	1.077	0.671	0.966
0.5	1.0	1.118	0.662	0.969
0.6	1.0	1.166	0.650	0.973
0.7	1.0	1.220	0.641	0.977
0.8	1.0	1.280	0.635	0.980
0.9	1.0	1.345	0.632	0.981
1.0	1.0	1.414	0.632	0.982

$d_{rms} = \sqrt{\sigma_x^2 + \sigma_y^2}$ when σ_x and σ_y are at right angles to each other.

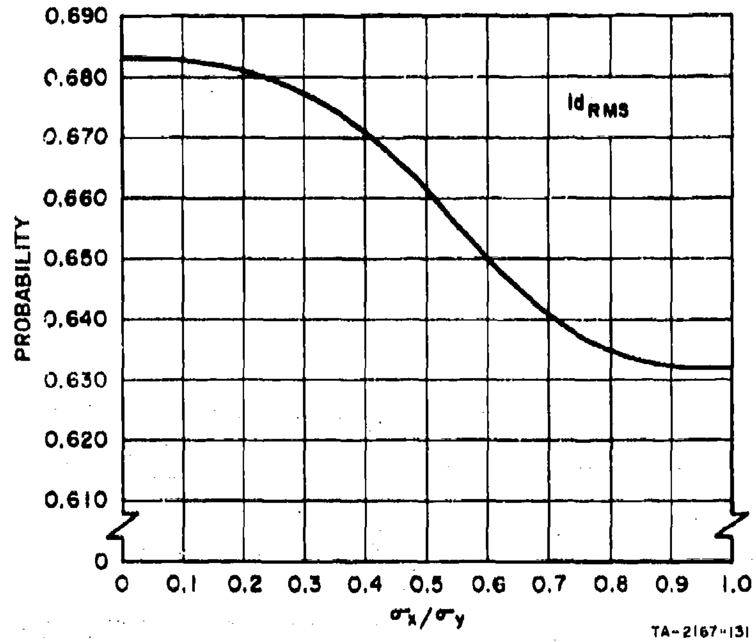

FIG. A-6 VARIATION IN d_{rms} WITH ELLIPTICITY — 1 d_{rms}

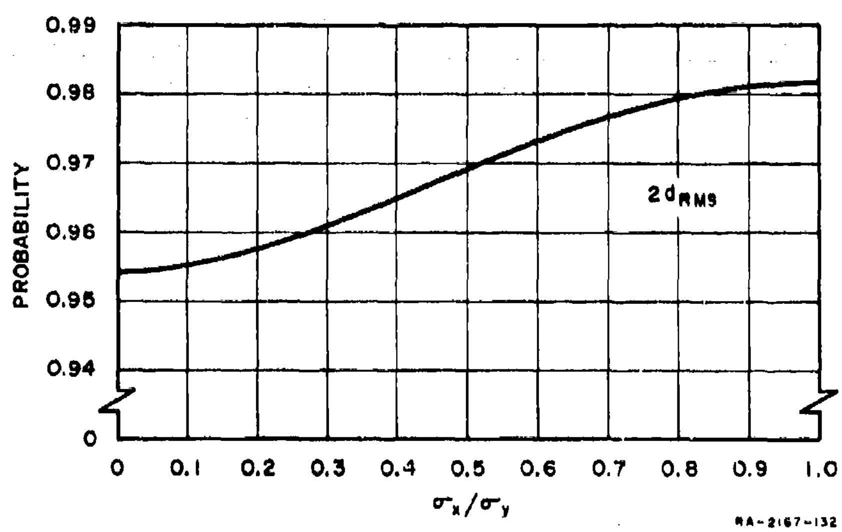

FIG. A-7 VARIATION IN d_{rms} WITH ELLIPTICITY — 2 d_{rms}

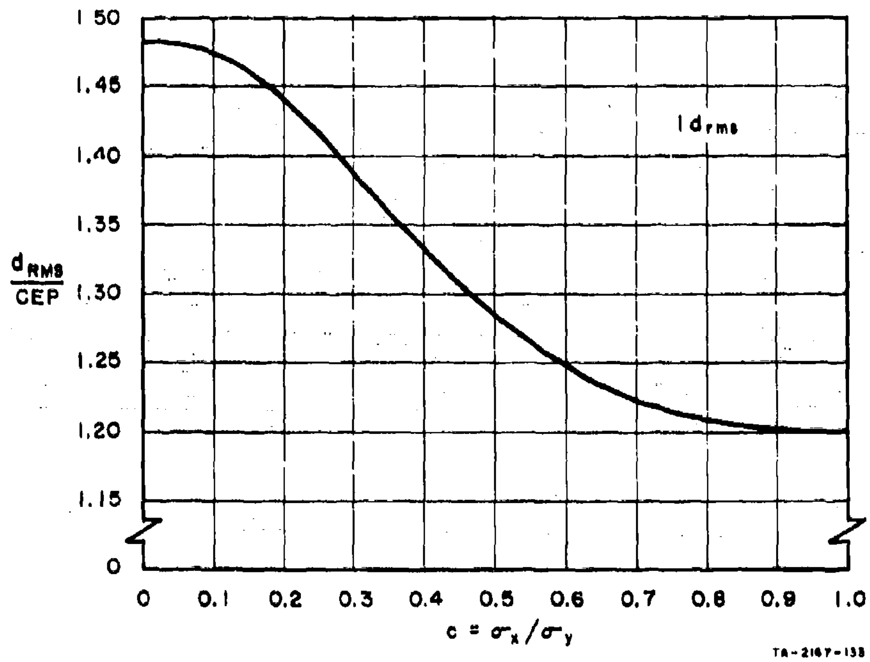

FIG. A-8 d_{rms}/CEP vs. ELLIPTICITY — 1 d_{rms}

APPENDIX B

DERIVATION OF METHOD 2 FORMULAS

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APPENDIX B

DERIVATION OF METHOD 2 FORMULAS

1. INTRODUCTION

In the body of this memorandum it was shown that the most general case of analyzing the accuracy of a positioning system resulted in the consideration of basic input error vectors intersecting at any angle. Graphs, nomograms, and tables were shown that may be used to obtain solutions to problems of accuracy determination where the lines of position do not intersect at right angles. Two methods of solutions were given there. This appendix presents a complete derivation of the formulas to obtain the standard deviations along the major and minor axes of an error ellipse. These functions are then used as the inputs to Harter's method of determination of elliptical probabilities described at the end of the Method 2 discussion.

The presentation of the Method 2 analysis is given ahead of that for Method 1 (see Appendix D) because of the detailed discussion developed during this analysis, which is basic in philosophy to both methods. Also, some of the formulas developed in this appendix are useful in derivations of some of the intermediate Method 1 solutions.

Nonorthogonal bivariate distributions arise when the lines of position do not intersect at right angles. Such a condition is the usual condition whatever the type of navigational system used—hyperbolic, trilateration, or conventional navigation with chronometer and sextant. Given two lines of position each with its own standard deviation, the problem is to determine the probability that the measured position is within a certain distance of the true position. Also of interest is the inverse problem, to determine the radius of a circle around the measured position within which the navigator knows, with a given probability, his true position lies.

An explicit solution for the integral which must be evaluated to solve these two problems is not possible, but the integrals can be

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evaluated by a digital computer. The mathematical literature seems to contain neither tabulated values of the integrals for the nonorthogonal case nor analysis of nonorthogonal bivariate normal distributions. Material on orthogonal bivariate normal distributions is, however, comparatively abundant (Refs. 1-8). In order to avoid the necessity for performing many integrations by quadratures, a technique has been found which permits existing tables of the bivariate normal distribution to be used to solve the problems listed above. This appendix will derive the necessary equations for obtaining σ_x and σ_y. The use of these functions as part of Method 2 has already been described. The following appendix indicates how these same functions may be used, after calculation of some auxiliary functions, to obtain solutions from available tables other than those already illustrated in the body of the memorandum.

2. STATEMENT OF THE PROBLEM

Consider a general point P whose position is determined by measuring its distances, r_1 and r_2 respectively, from two points of known location, M and S (Fig. B-1). M and S might correspond to the master and slave stations of a trilateration system. They might also correspond to two points of known location to which the ranges r_1 and r_2 are measured by a ranging system such as a radar, a laser, or an optical rangefinder. M and S might be points on the surface of the earth directly beneath two stars that are being used to determine position by conventional navigation techniques, using sextant and chronometer. The measurements of distance are assumed to be normally distributed with standard deviations σ_1 and σ_2. In Fig. 1 concentric circles of $r_1 + \sigma_1$, r_1, and $r_1 - \sigma_1$ about M and $r_2 + \sigma_2$, r_2, and $r_2 - \sigma_2$ about S have been drawn to suggest the normal distribution of the two measurements of range. If the standard deviations in range are very small in comparison with their respective ranges, the concentric circles can be assumed to be straight lines as in Fig. B-2. In this illustration, coordinate axes u_1 and u_2 are defined perpendicular to l_1 and l_2 respectively, which represent the true lines of position correspondin to circles of radius r_1 and r_2 in Fig. B-1. Other lines of position have been added at distances σ_1 and σ_2 respectively from l_1 and l_2. With no loss in generality, the measured lines of position could be assumed to be

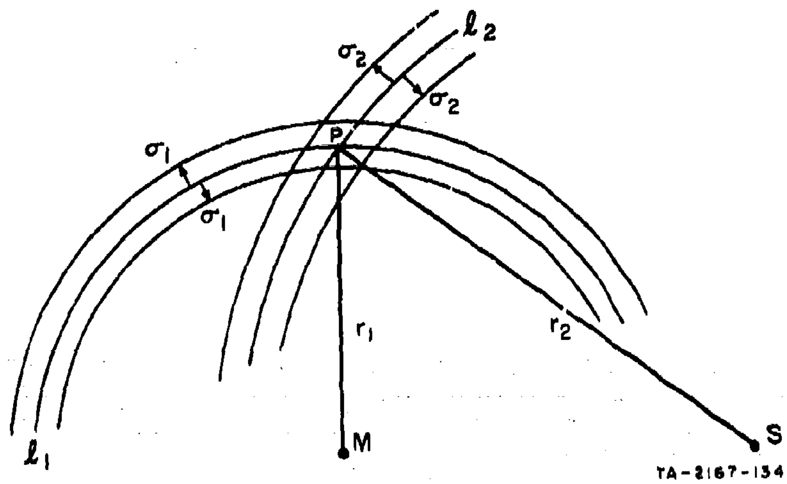

FIG. B-1 INTERSECTION OF TWO LINES OF POSITION

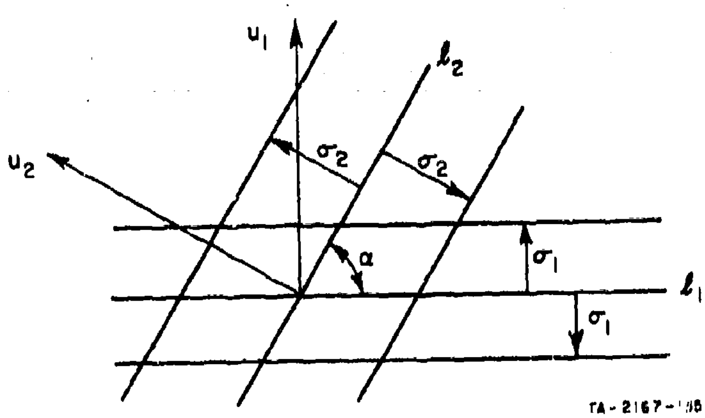

FIG. B-2 EXPANDED VIEW OF INTERSECTION

produced by conventional navigational techniques using sextant and theodolite, by a trilateration system, or by a hyperbolic navigation system. Random errors in measurement may cause a measured line of position to be displaced perpendicular to itself, and the probability that a measured line of position l_{1_m} will fall with a zone of width du_1 at a distance u_1 from the true line of position l_1, is given by

$$p_1 dy_1 = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{u_1^2}{2\sigma_1^2}} \qquad (1)$$

Similarly, the probability that the other measured line of position l_{2_m} will fall within a zone of width du_2 at a distance u_2 from the true line of position l_2 is given by

$$p_2 du_2 = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{u_2^2}{2\sigma_2^2}} du_2 \qquad (2)$$

Note that u_1 and u_2 are measured along the u_1 and u_2 axes, which in general are not orthogonal except under special conditions. The probability that the measured position will fall within an element of area du_1, du_2 centered on the point (u_1, u_2) is given by

$$P_{dA}\, du_1 du_2 = p_1 p_2 du_1 du_2 \qquad (3)$$

The probability that the measured position falls within an area of any size and shape, A, is found by integrating the above equation over that area.

$$P_A = \iint_A p_1 p_2 du_1 du_2 \qquad (4)$$

Therefore, in order to find the probability that the measured position lies within a distance r from the true position, the above integral must be evaluated over a circular area with radius r:

$$P_A = \frac{1}{2\pi\sigma_1\sigma_2} \iint_{A=\pi r^2} e^{-\frac{1}{2}\left(\frac{u_1^2}{\sigma_1^2} + \frac{u_2^2}{\sigma_2^2}\right)} du_1 du_2 \tag{5}$$

The inverse problem, to find the radius of a circle centered on the true position within which a measured position has a given probability of falling, is solved by fixing P_A in the above equation and solving for r. Next we shall show how existing tables can be used to evaluate the above nonorthogonal bivariate probability integral.

3. EVALUATION OF THE PROBABILITY INTEGRAL

In the above integral u_1 and u_2 are stochastically independent nonorthogonal variables. The probability integral will be solved by first converting to a new orthogonal coordinate system with normally distributed variables. However, these new orthogonal variables will not lie along the major and minor axes of the ellipse. Hence the equation of the ellipse, expressed in terms of the new coordinate axes of z_1 and z_2 will contain a cross-product term in $z_1 z_2$. Since such terms are inconvenient for ready algebraic manipulation, a second coordinate transformation, consisting of a simple rotation, will be employed to remove this cross-product term. The final answers thus obtained will be the standard deviations along the major and minor axes of the ellipse.

Neither of the two coordinate transformations alters the shape of the bivariate probability distribution. After the second transformation, the resulting probability integral can be evaluated by existing tables, such as those given in Refs. 1 through 4

The first coordinate conversion, illustrated in Figs. B-3, B-4, and B-5, is from u_1, u_2 coordinates to z_1, z_2 coordinates. The latter two axes are orthogonal with z_2 coincident with u_1. The former two axes are not orthogonal, and the y_1 and y_2 coordinates of the point are found by drawing two lines through the point parallel to the u_1

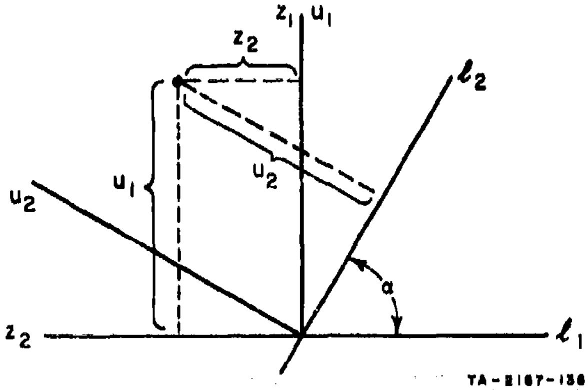

FIG. B-3 FIRST COORDINATE CONVERSION

and u_2 axes. Thus u_1 and u_2 coordinates are then found by measuring the distance from where these lines intersect the u_1 and u_2 axes to the origin of the coordinate system. By elementary analytical geometry,

$$y_1 = z_1 \qquad u_1 = z_1 \tag{6}$$

and

$$u_2 = \frac{kz_2 + z_1}{(1 + k^2)^{1/2}} \tag{7}$$

where k is a constant describing the slope of line of position l_2.

$$k = \tan \alpha \tag{8}$$

where α is the angle between the two lines of position.

It is necessary when converting from one coordinate system to another to replace the old element of area by a new element of area multiplied by the Jacobian of the old variables with respect to the new variables. That is

$$du_1 du_2 = J \left(\frac{u_1, u_2}{z_1, z_2}\right) dz_1 dz_2 \tag{9}$$

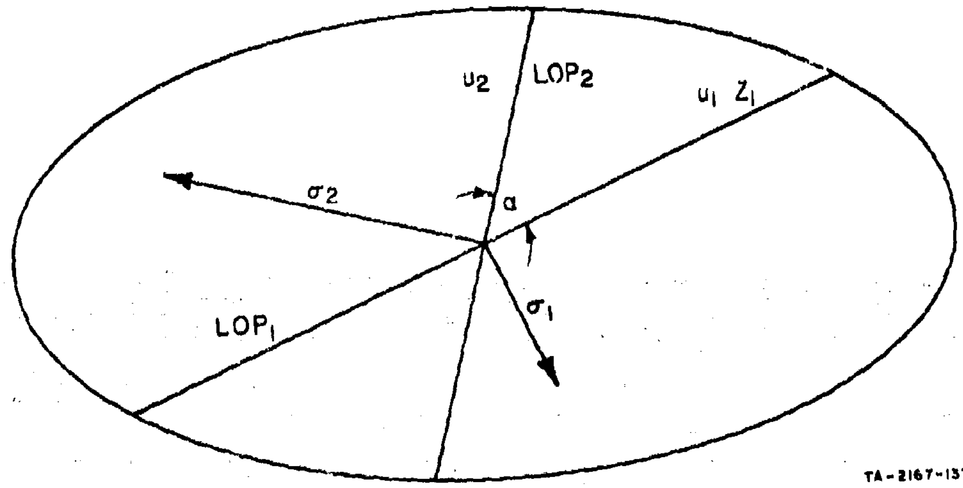

FIG. B-4 GIVEN CONDITIONS — ERROR ELLIPSE

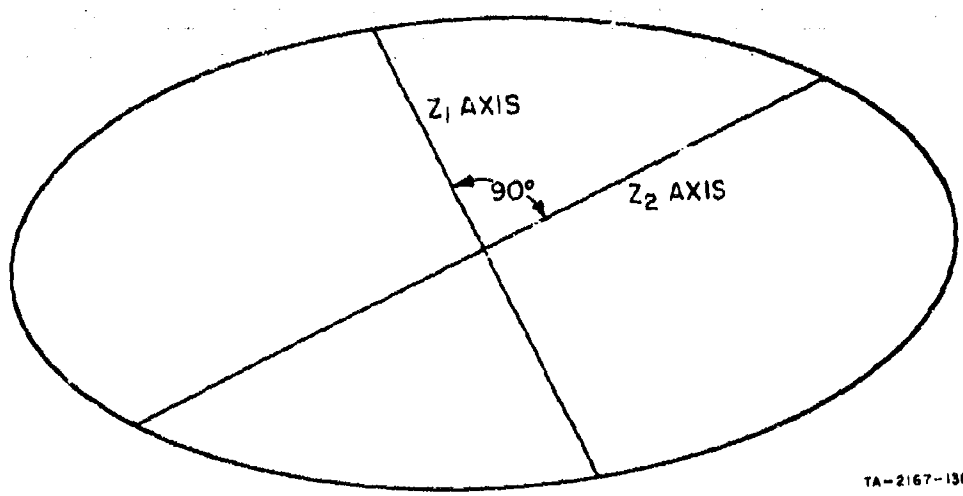

FIG. B-5 FIRST AXIS TRANSFORMATION

By definition

$$J\left(\frac{u_1, u_2}{z_1, z_2}\right) = \frac{\partial(u_1, u_2)}{\partial(z_1, z_2)} = \frac{\partial u_1}{\partial z_1}\frac{\partial u_2}{\partial z_2} - \frac{\partial u_1}{\partial z_2}\frac{\partial u_2}{\partial z_1} \tag{10}$$

When the indicated partial derivatives are evaluated by partial differentiation of equations (6) and (7),

$$J\left(\frac{u_1, u_2}{z_1, z_2}\right) = \frac{k}{(1 + k^2)^{1/2}} \sin \alpha \tag{11}$$

Therefore,

$$du_1 du_2 = \frac{k}{(1 + k^2)^{1/2}} dz_1 dz_2 \tag{12}$$

By substituting equations (6), (7), and (12) into equation (5)

$$P_A = \frac{1}{2\pi\sigma_1\sigma_2} \iint_A e^{-\frac{1}{2}\left(\frac{u_1}{\sigma_1^2}\right)+\left(\frac{u_2}{\sigma_2^2}\right)} du_1 du_2$$

$$= \frac{1}{2\pi(1 + k^2)^{1/2}} \iint_A e^{-\frac{1}{2}\left\{z_1^2\left[\frac{1}{\sigma_1^2} + \frac{1}{\sigma_2^2(1+k^2)}\right] + \frac{2kz_1z_2}{\sigma_2^2(1+k^2)} + \frac{k^2z_2^2}{\sigma_2^2(1+k^2)}\right\}} dz_1 dz_2 \tag{13}$$

The change of variables has not disturbed the location of any point, and therefore the circular area over which the integration is to be carried out is unchanged.

Examination of the exponent shows a cross-product term in z_1z_2. Only if k were infinite, corresponding to the special case in which the u_1 and u_2 axes were perpendicular, would this term disappear. In order to remove this term, the following transformation, which corresponds to a rotation through an angle θ, will be applied:

$$z_1 = x \cos \theta - y \sin \theta \tag{14}$$

$$z_L = x \sin\theta + y \cos\theta \tag{15}$$

The Jacobian of this transformation is one, so $dxdy$ can be substituted for dz_1dz_2. By substituting the above equations into the exponent of 3 in equation (13),

$$z_1^2\left[\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2(1+k^2)}\right]+\frac{2kz_1z_2}{\sigma_2^2(1+k^2)}+\frac{k^2z_2^2}{\sigma_2^2(1+k^2)} = x^2\left\{\frac{1}{\sigma_1^2\sigma_2^2(1+k^2)}\left[(\sigma_1^2+\sigma_2^2+k^2\sigma_2^2)\cos^2\theta + 2k\sigma_1^2\sin\theta\cos\theta + k^2\sigma_1^2\sin^2\theta\right]\right\}$$
$$+ xy\left\{\frac{1}{\sigma_1^2\sigma_2^2(1+k^2)}\left[2k\sigma_1^2(\cos^2\theta - \sin^2\theta) - (\sigma_1^2 - k^2\sigma_1^2 + \sigma_2^2 + k^2\sigma_2^2)(2\sin\theta\cos\theta)\right]\right\}$$
$$+ y^2\left\{\frac{1}{\sigma_1^2\sigma_2^2(1+k^2)}\left[(\sigma_1^2+\sigma_2^2+k^2\sigma_2^2)\sin^2\theta - 2k\sigma_1^2\sin\theta\cos\theta + k^2\sigma_1^2\cos^2\theta\right]\right\} \tag{16}$$

In order to eliminate the cross-product term in xy, the coefficient of the xy term in the preceding equation must equal zero. This occurs when

$$\cot 2\theta = \frac{\sigma_1^2(1-k^2)+\sigma_2^2(1+k^2)}{2k\sigma_1^2} \tag{17}$$

$$= \frac{\sigma_1^2(1-\tan^2\alpha)+\sigma_2^2(1+\tan^2\alpha)}{2\tan\alpha\sigma_1^2} \tag{17a}$$

An alternative form is

$$\cot 2\theta = \frac{\sigma_1^2\cos^2\alpha+\sigma_2^2}{\sin 2\alpha\sigma_1^2} \tag{17b}$$

Under this condition x and y become the semi-major and semi-minor axes of the ellipse. (See Fig. B-6.)

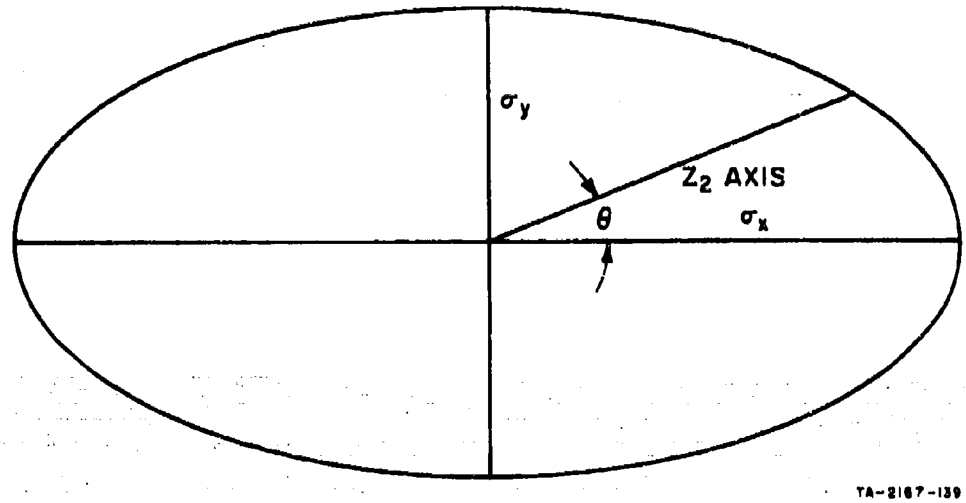

FIG. B-6 SECOND AXIS TRANSFORMATION

In the special case when σ_1 and σ_2 are equal,

$$\tan 2\theta = k \tag{18}$$

The slope of line of position l_2 with respect to line of position l_1 is k, and if the angle included by the two lines of position is called α, then

$$\tan \alpha = k \tag{19}$$

$$\theta = \frac{\alpha}{2} \tag{20}$$

Therefore, when the standard deviations of the distributions of the two lines of position are equal, the major and minor axes of the contour lines of equal probability on the nonorthogonal bivariate probability distribution are midway between the lines of position. (Compare Method 1.)

With the cross-product term eliminated by rotation through the angle θ defined by equation (17) the desired probability is given by

$$P_A = \frac{k}{2\pi(1+k^2)^{1/2}\sigma_1\sigma_2}\iint_A e^{-\frac{1}{2}\left(\frac{x^2}{\sigma_x^2}+\frac{y^2}{\sigma_y^2}\right)}\,dx\,dy \qquad (21)$$

From equation (16), the variances σ_x^2 and σ_y^2 may be found from

$$\frac{1}{\sigma_x^2} = \frac{1}{\sigma_1^2\sigma_2^2(1+k^2)}[(\sigma_2^2+\sigma_1^2+k^2\sigma_2^2)\cos^2\theta + 2k\sigma_1^2\sin\theta\cos\theta + k^2\sigma_1^2\sin^2\theta] \qquad (22)$$

and

$$\frac{1}{\sigma_y^2} = \frac{1}{\sigma_1^2\sigma_2^2(1+k^2)}[(\sigma_2^2+\sigma_1^2+k^2\sigma_2^2)\sin^2\theta - 2k\sigma_1^2\sin\theta\cos\theta + k^2\sigma_1^2\cos^2\theta] \qquad (23)$$

Values of σ_x^2 and σ_y^2 are needed so that equation (21) may be solved by means of existing tables of the bivariate normal distribution whose use will be discussed in the next appendix. Once θ has been found using equation (17) the preceding two equations can be used to calculate σ_x^2 and σ_y^2. It might be objected that this is a laborious process. Fortunately it is possible to obtain equations for σ_x^2 and σ_y^2 as simpler functions of σ_1, σ_2, and α. By adding equations (22) and (23), thus obviating the need for the auxiliary computation of the angle θ, we obtain

$$\frac{1}{\sigma_x^2}+\frac{1}{\sigma_y^2} = \frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2} \qquad (24)$$

Also from equations (22) and (23)

$$\begin{aligned}\sigma_x^2\sigma_y^2 = \{\sigma_1^2\sigma_2^2(1+k^2)\}^2 \{&[(\sigma_2^2+\sigma_1^2+k^2\sigma_2^2)^2 - 4k^2\sigma_1^4 + k^4\sigma_1^4]\sin^2\theta\cos^2\theta \\ &+ [k^3\sigma_1^4 - k\sigma_1^2(\sigma_2^2+\sigma_1^2+k^2\sigma_2^2)]\sin\theta\cos\theta(\cos^2\theta-\sin^2\theta) \\ &+ k^2\sigma_1^2(\sigma_2^2+\sigma_1^2+k^2\sigma_2^2)(\sin^4\theta+\cos^4\theta)\}^{-1}\end{aligned} \qquad (25)$$

From equation (17) it follows that

$$\sin 2\theta = \frac{2k\sigma_1^2}{\left\{[2k\sigma_1^2]^2 + [\sigma_1^2(1-k^2)+\sigma_2^2(1+k^2)]^2\right\}^{1/2}} \qquad (26)$$

$$\cos 2\theta = \frac{\sigma_1^2(1 - k^2) + \sigma_2^2(1 + k^2)}{\{[2k\sigma_1^2]^2 + [\sigma_1^2(1 - k^2) + \sigma_2^2(1 + k^2)]^2\}^{1/2}} \tag{27}$$

By these equations together with the following trigonometric identities

$$2 \sin \theta \cos \theta = \sin 2\theta \tag{28}$$

$$\cos^2 \theta - \sin^2 \theta = \cos 2\theta \tag{29}$$

and

$$\sin^4 \theta + \cos^4 \theta = 1 - 2 \sin^2 \theta \cos^2 \theta \tag{30}$$

equation (25) can be simplified to

$$\sigma_x^2\sigma_y^2 = \frac{(1 + k^2)}{k^2}\sigma_1^2\sigma_2^2 \tag{31}$$

Since

$$k = \tan \alpha \tag{32}$$

where α is the angle between the two lines of position,

$$\frac{k^2}{1 + k^2} = \sin^2 \alpha \tag{33}$$

and equation (31) may be rewritten as

$$\sigma_x^2\sigma_y^2 = \frac{\sigma_1^2\sigma_2^2}{\sin^2 \alpha} \tag{34}$$

When equation (34) is combined with equation (24),

$$\sigma_x^2 + \sigma_y^2 = \frac{\sigma_1^2 + \sigma_2^2}{\sin^2 \alpha} \tag{35}$$

Equations (34) and (35) can be solved for σ_x^2 and σ_y^2

$$\sigma_x^2 = \frac{1}{2 \sin^2 x}[\sigma_1^2 + \sigma_2^2 + \sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4(\sin^2 \alpha)\sigma_1^2\sigma_2^2}] \tag{36}$$

$$\sigma_y^2 = \frac{1}{2\sin^2\alpha}\left[\sigma_1^2 + \sigma_2^2 - \sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4(\sin^2\alpha)\sigma_1^2\sigma_2^2}\right] \qquad (37)$$

With the aid of tables of squares and squares of sines of angles (Ref. 10, for example), these equations may be solved to determine numerical values. It is to be noted that the equations are in terms of variances; to obtain standard deviations as desired for some calculations square roots must be taken. It is also to be noted that the formulas for the two desired functions differ only in the sign before the radical sign. Thus in numerical calculation the solution of one expression provides all the numbers required for the second.

For the special case when $\sigma_1 = \sigma_2$ equations (36) and (37) may be greatly simplified as follows:

$$\sigma_x^2 = \frac{1}{2\sin^2\alpha}\{2\sigma^2 + \sqrt{4\sigma^4 - 4\sin^2\alpha\sigma^4}\} \qquad (38)$$

$$= \frac{1}{2\sin^2\alpha}\{2\sigma^2 + 2\sigma^2\sqrt{1 - \sin^2\alpha}\} \qquad (39)$$

$$= \frac{1}{2\sin^2\alpha}(2\sigma^2[1 + \cos\alpha]) \qquad (40)$$

$$= \frac{1 + \cos\alpha}{\sin^2\alpha}\sigma^2 \qquad (41)$$

$$\sigma_x = \frac{\sqrt{1 + \cos\alpha}}{\sin\alpha}\sigma \qquad (42)$$

Since

$$\cos\frac{\alpha}{2} = \sqrt{\frac{1 + \cos\alpha}{2}} \qquad (43)$$

$$\sin\alpha = \sin 2(\alpha/2) \qquad (44)$$

Then

$$\sigma_x = \frac{\sigma\sqrt{2}}{2\sin\frac{\alpha}{2}} \qquad (45)$$

and similarly

$$\sigma_y = \frac{\sigma\sqrt{2}}{2 \cos \frac{\alpha}{2}} \tag{46}$$

The ratio σ_y/σ_x (= c of Method 2, Section 2) is useful in further conversion of the error ellipse to circles of equivalent probability and may be readily seen to be

$$\frac{\sigma_y}{\sigma_x} = c = \tan 1/2\ \alpha \qquad \text{when } \sigma_1 = \sigma_2 \tag{47}$$

4. ALTERNATIVE FORMULAS

Reference 9 lists the following formulas for σ_x and σ_y

$$\sigma_x^2 = \frac{1}{2}(\sigma_1^2 + \sigma_2^2) \csc^2 \alpha + \sigma_1 \cot \alpha \csc 2\theta$$

$$\sigma_y^2 = \frac{1}{2}(\sigma_1^2 + \sigma_2^2) \csc^2 \alpha - \sigma_1 \cot \alpha \csc 2\theta$$

$$\tan 2\theta = \frac{\sigma_1^2 \sin 2\alpha}{\sigma_1^2 \cos 2\alpha + \sigma_2^2}$$

θ is the angle between σ_1 and σ_x as in the foregoing analysis.

Although the expressions of equations (36) and (37) initially appear lengthy, experience has shown that they are well suited to computation. They have the advantage, too, not possessed by the alternative formulas, of not requiring auxiliary computation of additional functions before the given values may be entered into the formulas. Equations (36) and (37) utilize only the basic input data—the errors along the given axes and the angle between these axes. Available handbooks of mathematical tables (Ref. 10, for example) contain tables of squares and also of the squares of sines of angles so that the numerical values needed may be readily obtained. Although a desk calculator is handy, slide rule computations are not difficult and are of sufficient accuracy for systems evaluation.

APPENDIX C

EXAMPLES OF THE USE OF TABLES OF THE BIVARIATE PROBABILITY DISTRIBUTION

APPENDIX C

EXAMPLES OF THE USE OF TABLES OF THE BIVARIATE PROBABILITY DISTRIBUTION

1. INTRODUCTION

Reference 1 contains the most extensive set of tables applicable to bivariate normal distributions. The use of these tables requires the calculation of certain auxiliary functions, which are described in this section. A number of problems illustrating the use of these functions and tables follow.

Let $P_r\{a_1u_1^2 + a_2u_2^2 \leq t\}$ denote the probability that a randomly chosen point will fall inside an ellipse whose equation is

$$a_1u_1^2 + a_2u_2^2 = t \tag{1}$$

where u_1 and u_2 are stochastically independent normally distributed variables with means equal to zero and standard deviations equal to one. By definition,

$$a_1 + a_2 = 1 \tag{2}$$

then

$$P_2(a_1, a_2; t) = P_r\{a_1u_1^2 + a_2u_2^2 \leq t\} \tag{3}$$

and

$$P_2(a_1, a_2; t) = \frac{1}{2\pi\sigma_{u1}\sigma_{u2}} \iint_{a_1u_1^2+a_2u_2^2=t} e^{-\frac{1}{2}\left(\frac{u_1^2}{\sigma_{u1}^2} + \frac{u_2^2}{\sigma_{u2}^2}\right)} du_1 du_2 \tag{4}$$

where $\sigma_{u1} = \sigma_{u2} = 1$.

Tables of $P_2(a_1, a_2; t)$ are given in Refs. 1, 2 and 3. Reference 1 gives 120 values of $P_2(a_1, a_2; t)$ to four significant figures, with

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$t = 0.1(0.1)\ 1.0(0.5)\ 2.0(1.0)\ 5.0$. (This means that t has intervals of 0.1 between 0.1 and 1.0, and 0.5 between 1.0 and 2.0 etc.) and with $a_2 = 0.5,\ 0.6,\ 0.7,\ 0.8,\ 0.9,\ 0.95,\ 0.99$ and 1.00. Reference 2 gives 149 values of $P_2(a_1, a_2; t)$ to five significant figures, with $t = 0.1(0.1)\ 1.0(0.5)\ 2.0(1.0)\ 5.0$ and with $a_2 = 0.5,\ 0.6,\ 2/3,\ 0.7,\ 0.8,\ 0.875,\ 0.9,\ 0.95$ and 0.99. Another table in Ref. 3 gives 150 values of the inverse of a quadratic form in two dimensions, *i.e.*, the value of t which causes $P_2(a_1, a_2; t)$ to have a certain value. The values of a_2 are the same as in the preceding table, and $P = 0.05(0.05)\ 0.30(0.10)\ 0.70(0.05)\ 0.95$. Reference 3 gives 2881 values of $P_2(a_1, a_2; t)$ to eight significant figures with $t = 0.005(0.005)\ 0.10(0.01)\ 1.00(0.02)\ 2.50(0.025)\ 3.50(0.05)\ 5.00(0.25)\ 6.00(0.50)\ 7.00(1.00)\ 10.00$ and $a_2 = 0.5,\ 0.6,\ 2/3,\ 0.7,\ 0.75,\ 0.8,\ 0.875,\ 0.9,\ 0.95,\ 0.99$ and 1.00.

These tables can still be used even if the two variables do not have unit standard deviations. Consider two normally distributed variables x_1 and x_2 with standard deviations σ_{x1} and σ_{x2} respectively. If the bivariate probability integral is evaluated over the area inside an ellipse

$$b_1 x_1^2 + b_2 x_2^2 = r^2 \tag{5}$$

then

$$P_r\{b_1 x_1^2 + b_2 x_2^2 \leq r^2\} = \frac{1}{2\pi\sigma_{x1}\sigma_{x2}} \iint\limits_{b_1 x_1^2 + b_2 x_2^2 = r^2} e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_{x1}^2} + \frac{x_2^2}{\sigma_{x2}^2}\right)} dx_1 dx_2 \tag{6}$$

In order that

$$P_r\{b_1 x_1^2 + b_2 x_2^2 \leq r^2\} = P_r\{a_1 u_1^2 + a_2 u_2^2 \leq t\} = P_2(a_1, a_2; t) \tag{7}$$

it is necessary that

$$\frac{b_1 x_1^2}{a_1 u_1^2} = \frac{b_2 x_2^2}{a_2 u_2^2} = \frac{r^2}{t} = C \tag{8}$$

where C is a constant.

Since x_1, x_2, u_1, and u_2 are all normally distributed

$$\frac{x_1^2}{\sigma_{x1}^2} = \frac{u_1^2}{\sigma_{u1}^2} \tag{9}$$

and

$$\frac{x_2^2}{\sigma_{x2}^2} = \frac{u_2^2}{\sigma_{u2}^2} \quad . \tag{10}$$

Since σ_{u1} and σ_{u2} equal one,

$$u_1^2 = \frac{x_1^2}{\sigma_{x1}^2} \tag{11}$$

and

$$u_2^2 = \frac{x_2^2}{\sigma_{x2}^2} \quad . \tag{12}$$

When these values are substituted into equation (10)

$$a_1 = \frac{b_1\sigma_{x1}^2}{C} \tag{13}$$

$$a_2 = \frac{b_2\sigma_{x2}^2}{C} \tag{14}$$

and

$$t = \frac{r^2}{C} \tag{15}$$

Since by definition the sum of a_1 and a_2 is one,

$$\frac{b_1\sigma_{x1}^2}{C} + \frac{b_2\sigma_{x2}^2}{C} = 1 \tag{16}$$

or

$$C = b_1\sigma_{x1}^2 + b_2\sigma_{x2}^2 \tag{17}$$

Therefore

$$P_2(a_1, a_2; t) = \frac{1}{2\pi\sigma_{x1}\sigma_{x2}} \iint_{b_1x_1^2+b_2x_2^2=r^2} e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_{x1}^2}+\frac{x_2^2}{\sigma_{x2}^2}\right)} dx_1 dx_2 \tag{18}$$

where a_1, a_2 and t as functions of b_1, b_2, σ_{x1}^2, σ_{x2}^2 and r^2 are given by equations (13), (14), (15), and (16). These equations disagree with the ones given in Ref. 2; as the above derivation shows, Ref. 2 is wrong.

When equation (20) is substituted into equation (21) of Appendix B,

$$P_A = \frac{k\sigma_{x1}\sigma_{x2}}{\sigma_1^2\sigma_2(1+k^2)^{1/2}} P_2(a_1, a_2; t) \tag{19}$$

But from equation (31), Appendix B,

$$\sigma_{x1}\sigma_{x2} = \frac{(1+k^2)^{1/2}}{k} \sigma_1\sigma_2 \tag{20}$$

so

$$P_A = P_2(a_1, a_2; t) \tag{21}$$

Therefore, in the general case, P_A, the probability of being within an ellipse, $b_1x_1^2 + b_2x_2^2 = r^2$, can be read directly from tables of $P_2(a_1, a_2; t)$ after a_1, a_2, and t have been calculated means of equations (36) and (37) of Appendix B and (13), (14), (15), and (16). In the special case with which this memorandum is concerned, finding the probability that the true position is within a distance r of the indicated position, the calculations are still simpler. The ellipse becomes a circle, $b_1x_1^2 + b_2x_2^2 = r^2$, and

$$b_1 = b_2 = 1 \tag{22}$$

By substitution using equations (35) of Appendix B and

$$C = \frac{\sigma_1^2 + \sigma_2^2}{\sin^2 \alpha} \tag{23}$$

Equations (13) to (15) simplify to

$$a_1 = \frac{1}{2}\left\{1 + \left[1 - \frac{4(\sin^2\alpha)\sigma_1^2\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2}\right]^{1/2}\right\} \tag{24}$$

$$a_2 = \frac{1}{2}\left\{1 - \left[1 - \frac{4(\sin^2\alpha)\sigma_1^2\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2}\right]^{1/2}\right\} \tag{25}$$

$$t = \frac{r^2 \sin^2\alpha}{\sigma_1^2 + \sigma_2^2} \tag{26}$$

2. PROBLEMS

Table C-1 summarizes the values of the given quantities in the different problems and the resulting probabilities $P_2(a_1, a_2; t)$. Where possible the problems have been checked by use of different tables. As a convenience to the reader, Tables C-2, C-3, and C-4, abstracted from Ref. 2, are included at the end of this appendix so that the results presented here can be verified. Table C-2 is a brief table of $P_2(a_1, a_2; t)$.

Table C-1

SUMMARY OF PROBLEMS

PROBLEMS	α	σ_1	σ_2	b_1	b_2	r	$P_2(a_1, a_2; t)$
1	90°	1	1	1	1	1	0.39347
2	90°	1	1	1	1	2	0.86466
3	90°	2	2	1	1	2	0.39347
4	90°	1	1	1	2	2	0.74244
5	90°	1	2	1	1	1	0.21529
6	90°	1	2	1	1	2	0.59009
7	90°	1	2	4	1	2	0.39347
8	90°	1	2	1	4	2	0.32623
9	60°	1	1	1	1	1	0.34230
10	60°	1	1	1	1	$\sqrt{2/3}$	0.24601
11	60°	1	1	1	1	$\sqrt{2}$	0.55620
12	60°	1	1	1/2	3/2	1	0.39347
13	60°	1	1	3/2	1/2	1	0.34945
14	65°	700 ft	900 ft	1	1	1520 ft	0.77624
15	65°	700 ft	900 ft	1	1	3040 ft	0.99404

The more extensive table of $P_2(a_1, a_2; t)$ in Ref. 3 was actually used in working these problems because it has more entries, and the interval for interpolation is smaller. Table 2 can be used to integrate (1) a circular probability distribution over a circular area, (2) a circular probability distribution over an elliptical area, (3) an elliptical probability distribution over a circular area, or (4) an elliptical probability distribution over an elliptical area. A bivariate normal distribution such as

$$\frac{1}{2\pi\sigma_{x1}\sigma_{x2}} e^{-\frac{1}{2}\left(\frac{x_1^2}{\sigma_{x1}^2} + \frac{x_2^2}{\sigma_{x2}^2}\right)}$$

is called a circular distribution when $\sigma_{x1} = \sigma_{x2}$ because the contours of equal probability density are circular. If $\sigma_{x1} \neq \sigma_{x2}$ the lines of equal probability density are elliptical and the bivariate distribution is called an elliptical distribution. Table C-2 can be used to integrate a circular probability over a circular area, and Table C-3 can be used (1) to integrate a circular probability distribution over a circular area, or (2) at the expense of entering the table twice, to integrate a circular probability distribution over an elliptical area. Since the latter table occupies several pages, only those pages needed to work the problems in this memorandum have been included. As a check, several problems have been solved using Tables C-3 or C-4 or both as a check on the answer obtained from Table C-2. Unfortunately not all the problems could be so checked, because of the inherent limitations of Tables C-3 and C-4.

Problem (1)

Given

$$\alpha = 90°, \ \sigma_1 = \sigma_{x1} = 1, \ \sigma_2 = \sigma_{x2} = 1, \ b_1 = 1, \ b_2 = 1, \ r = 1$$

Find

$$P_2(a_1, a_2; t)$$

Since the lines of position are perpendicular ($\alpha = 90°$) and their standard deviations are equal, the probability distribution is circular. The area over which the integral is taken is likewise circular, ($b_1 x_1^2 + b_2 x_2^2 = r^2$ or $x_1^2 + x_2^2 = 1$) and the area has a radius equal to one standard deviation.

Table C-2

DISTRIBUTION OF A QUADRATIC FORM IN TWO DIMENSIONS

a_2, a_1	t = 0.1	0.2	0.3	0.4	0.5
.5 , .5	0.09516	0.18127	0.25918	0.32968	0.39347
.6 , .4	0.09693	0.18429	0.26304	0.33405	0.39809
2/3, 1/3	0.10033	0.19005	0.27033	0.34222	0.40664
.7 , .3	0.10288	0.19432	0.27568	0.34815	0.41278
.75 , .25	0.10814	0.20299	0.28637	0.35982	0.42468
.8 , .2	0.11581	0.21529	0.30112	0.37550	0.44023
.875, .125	0.13546	0.24481	0.33434	0.40866	0.47117
.9 , .1	0.14608	0.25941	0.34945	0.42257	0.48315
.95 , .05	0.18130	0.30018	0.38581	0.45206	0.50596
.99 , .01	0.23588	0.33838	0.41150	0.46966	0.51820

a_2, a_1	0.6	0.7	0.8	0.9	1.0
.5 , .5	0.45119	0.50341	0.55067	0.59343	0.63212
.6 , .4	0.45585	0.50797	0.55500	0.59746	0.63579
2/3, 1/3	0.46441	0.51625	0.56279	0.60462	0.64223
.7 , .3	0.47048	0.52205	0.56819	0.60952	0.64658
.75 , .25	0.48206	0.53294	0.57814	0.61839	0.65429
.8 , .2	0.49678	0.54640	0.59009	0.62872	0.66297
.875, .125	0.52435	0.57011	0.60985	0.64467	0.67540
.9 , .1	0.53423	0.57795	0.61587	0.64910	0.67848
.95 , .05	0.55133	0.59040	0.62460	0.65488	0.68192
.99 , .01	0.55982	0.59615	0.62827	0.65692	0.68266

a_2, a_1	1.5	2.0	3.0	4.0	5.0
.5 , .5	0.77687	0.86466	0.95021	0.98168	0.99326
.6 , .4	0.77849	0.86461	0.94871	0.98023	0.99227
2/3, 1/3	0.78108	0.86424	0.94600	0.97773	0.99057
.7 , .3	0.78262	0.86379	0.94412	0.97608	0.98946
.75 , .25	0.78491	0.86255	0.94066	0.97316	0.98752
.8 , .2	0.78664	0.86036	0.93653	0.96984	0.98531
.875, .125	0.78670	0.85500	0.92944	0.96436	0.98160
.9 , .1	0.78581	0.85274	0.92695	0.96244	0.98028
.95 , .05	0.78296	0.84784	0.92187	0.95851	0.97753
.99 , .01	0.78009	0.84374	0.91776	0.95531	

Table C-3

CRITICAL VALUES FOR THE CIRCULAR NORMAL DISTRIBUTION

P	R	P	R	P	R
0.01	0.1418	0.46	1.1101	0.91	2.1945
0.02	0.2010	0.47	1.1268	0.92	2.2475
0.03	0.2468	0.48	1.1436	0.93	2.3062
0.04	0.2857	0.49	1.1605	0.94	2.3721
0.05	0.3203	0.50	1.1774	0.95	2.4477
0.06	0.3518	0.51	1.1944	0.96	2.5373
0.07	0.3810	0.52	1.2116	0.97	2.6482
0.08	0.4084	0.53	1.2288	0.98	2.7971
0.09	0.4343	0.54	1.2462	0.99	3.0349
0.10	0.4590	0.55	1.2637	0.991	3.0694
0.11	0.4828	0.56	1.2814	0.992	3.1075
0.12	0.5056	0.57	1.2992	0.993	3.1502
0.13	0.5278	0.58	1.3172	0.994	3.1987
0.14	0.5492	0.59	1.3354	0.995	3.2552
0.15	0.5701	0.60	1.3537	0.996	3.3231
0.16	0.5905	0.61	1.3723	0.997	3.4086
0.17	0.6105	0.62	1.3911	0.998	3.5255
0.18	0.6300	0.63	1.4101	0.999	3.7169
0.19	0.6492	0.64	1.4294	0.9991	3.7452
0.20	0.6680	0.65	1.4490	0.9992	3.7765
0.21	0.6866	0.66	1.4689	0.9993	3.8117
0.22	0.7049	0.67	1.4891	0.9994	3.8519
0.23	0.7230	0.68	1.5096	0.9995	3.8989
0.24	0.7409	0.69	1.5305	0.9996	3.9558
0.25	0.7585	0.70	1.5518	0.9997	4.0278
0.26	0.7760	0.71	1.5735	0.9998	4.1273
0.27	0.7934	0.72	1.5956	0.9999	4.2919
0.28	0.8106	0.73	1.6182	0.99995	4.4505
0.29	0.8276	0.74	1.6414	0.99999	4.7985
0.30	0.8446	0.75	1.6651	0.999995	4.9409
0.31	0.8615	0.76	1.6894	0.999999	5.2565
0.32	0.8783	0.77	1.7145	0.9999995	5.3868
0.33	0.8950	0.78	1.7402	0.9999999	5.6777
0.34	0.9116	0.79	1.7667	0.99999995	5.7985
0.35	0.9282	0.80	1.7941	0.99999999	6.0697
0.36	0.9448	0.81	1.8225	0.999999995	6.1829
0.37	0.9613	0.82	1.8519	0.999999999	6.4379
0.38	0.9778	0.83	1.8825		
0.39	0.9943	0.84	1.9145		
0.40	1.0108	0.85	1.9479		
0.41	1.0273	0.86	1.9830		
0.42	1.0438	0.87	2.0200		
0.43	1.0603	0.88	2.0593		
0.44	1.0769	0.89	2.1011		
0.45	1.0935	0.90	2.1460		

Table C-4

OFFSET CIRCLE PROBABILITIES FOR THE CIRCULAR NORMAL DISTRIBUTION

$(r_4 - D)/\sigma$

D/σ	-1.9	-1.8	-1.7	-1.6	-1.5	-1.4	-1.3	-1.2	-1.1	-1.0	-.9	-.8	-.7	-.6	-.5	-.4	-.3	-.2	-.1	0
.0																				1,000
.1																			1,000	.995
.2																		1,000	.995	.981
.3																	1,000	.995	.981	.958
.4																1,000	.995	.982	.959	.929
.5															1,000	.996	.983	.961	.932	.896
.6														1,000	.996	.983	.963	.939	.901	.860
.7													1,000	.998	.984	.965	.939	.907	.868	.825
.8												1,000	.996	.986	.968	.943	.913	.877	.836	.791
.9											1,000	.997	.987	.970	.946	.920	.886	.848	.808	.760
1.0										1,000	.997	.988	.973	.952	.927	.896	.860	.821	.778	.733
1.1									1,000	.997	.989	.976	.957	.933	.905	.873	.836	.797	.754	.709
1.2								1,000	.998	.990	.978	.962	.940	.915	.885	.852	.815	.775	.733	.688
1.3							1,000	.998	.991	.981	.966	.947	.924	.897	.867	.833	.796	.756	.715	.671
1.4						1,000	.998	.992	.983	.970	.953	.933	.909	.881	.850	.816	.780	.741	.699	.656
1.5					1,000	.998	.993	.985	.974	.959	.941	.920	.895	.867	.836	.802	.766	.727	.686	.644
1.6				1,000	.999	.994	.987	.978	.965	.949	.930	.909	.883	.858	.824	.790	.754	.716	.675	.633
1.7			1,000	.999	.995	.989	.981	.970	.958	.940	.920	.898	.873	.845	.814	.780	.744	.706	.666	.624
1.8		1,000	.998	.996	.991	.984	.974	.963	.948	.931	.912	.889	.864	.836	.805	.771	.736	.697	.658	.616
1.9	1,000	.999	.997	.992	.986	.978	.968	.956	.941	.924	.904	.882	.856	.828	.797	.764	.728	.690	.651	.610
2.0	.999	.997	.994	.989	.982	.974	.963	.950	.936	.918	.898	.875	.850	.822	.791	.757	.722	.684	.644	.604
2.1	.998	.995	.991	.985	.978	.969	.959	.946	.930	.913	.893	.870	.844	.816	.785	.752	.716	.678	.639	.598
2.2	.996	.992	.988	.982	.975	.966	.955	.941	.926	.908	.888	.865	.839	.811	.780	.747	.711	.673	.634	.593
2.3	.994	.990	.985	.979	.972	.962	.951	.938	.922	.904	.884	.861	.835	.807	.776	.742	.708	.669	.630	.589
2.4	.992	.988	.983	.977	.969	.960	.948	.935	.919	.901	.880	.857	.831	.803	.771	.738	.702	.665	.626	.585
2.5	.991	.986	.981	.975	.967	.957	.946	.932	.916	.898	.877	.854	.828	.799	.768	.734	.699	.661	.622	.582
2.6	.989	.985	.980	.973	.965	.955	.943	.930	.914	.895	.874	.851	.825	.796	.765	.731	.696	.658	.619	.578
2.7	.988	.984	.978	.971	.963	.953	.941	.928	.911	.893	.872	.848	.822	.793	.762	.728	.692	.655	.616	.575
2.8	.987	.983	.977	.970	.962	.952	.940	.926	.909	.891	.869	.846	.819	.790	.759	.725	.689	.652	.613	.572
2.9	.986	.982	.976	.969	.960	.950	.938	.924	.907	.889	.867	.843	.817	.788	.756	.722	.687	.649	.610	.570
3.0	.985	.981	.975	.968	.959	.949	.937	.922	.906	.887	.865	.841	.815	.785	.754	.720	.684	.647	.608	.567
3.1	.985	.980	.974	.967	.958	.948	.935	.921	.904	.885	.863	.839	.813	.783	.752	.718	.682	.644	.605	.565
3.2	.984	.979	.973	.966	.957	.947	.934	.920	.903	.883	.862	.837	.811	.781	.750	.716	.680	.642	.603	.563
3.3	.984	.979	.973	.965	.956	.946	.933	.918	.901	.882	.860	.836	.809	.780	.748	.714	.678	.640	.601	.561
3.4	.983	.978	.972	.964	.955	.945	.932	.917	.900	.881	.859	.834	.807	.778	.746	.712	.676	.638	.599	.559
3.5	.983	.978	.971	.964	.955	.944	.931	.916	.899	.879	.857	.833	.806	.776	.744	.710	.674	.637	.598	.558
3.6	.982	.977	.971	.963	.954	.943	.930	.915	.898	.878	.856	.831	.804	.775	.743	.709	.673	.635	.596	.556
3.7	.982	.977	.970	.963	.953	.942	.929	.914	.897	.877	.855	.830	.803	.773	.741	.707	.671	.633	.594	.554
3.8	.982	.976	.970	.962	.953	.941	.928	.913	.896	.876	.854	.829	.802	.772	.740	.706	.670	.632	.593	.553
3.9	.981	.976	.969	.962	.952	.941	.928	.912	.895	.875	.853	.828	.800	.771	.739	.704	.668	.631	.591	.552

D/σ	.1	.2	.3	.4	.5	.6	.7	.8	.9	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
.0	.995	.980	.956	.923	.882	.835	.783	.726	.667	.607	.546	.487	.430	.375	.325	.278	.236	.198	.164	.135
.1	.980	.956	.923	.883	.836	.784	.727	.668	.608	.548	.489	.431	.377	.326	.280	.237	.200	.166	.137	.111
.2	.957	.925	.885	.838	.787	.731	.672	.613	.553	.494	.437	.383	.332	.285	.243	.204	.170	.141	.115	.093
.3	.926	.887	.842	.791	.736	.679	.620	.561	.502	.446	.392	.341	.294	.251	.212	.178	.148	.121	.099	.080
.4	.891	.847	.798	.744	.688	.630	.572	.514	.458	.404	.353	.306	.263	.223	.188	.157	.130	.106	.086	.069
.5	.853	.805	.754	.699	.643	.588	.529	.473	.420	.369	.322	.279	.237	.201	.169	.141	.116	.095	.077	.062
.6	.815	.765	.712	.658	.602	.546	.492	.439	.388	.340	.296	.255	.217	.184	.154	.128	.106	.086	.070	.056
.7	.778	.727	.674	.620	.566	.512	.460	.409	.361	.316	.274	.236	.201	.170	.143	.119	.098	.080	.064	.052
.8	.745	.693	.641	.588	.536	.484	.434	.385	.340	.297	.258	.221	.189	.160	.134	.111	.091	.075	.060	.049
.9	.712	.663	.612	.560	.510	.460	.412	.366	.322	.282	.244	.210	.179	.151	.127	.105	.087	.071	.057	.046
1.0	.685	.636	.587	.537	.488	.440	.394	.350	.308	.269	.233	.200	.171	.144	.121	.100	.083	.067	.055	.044
1.1	.662	.614	.566	.518	.470	.424	.379	.336	.296	.259	.224	.193	.164	.139	.116	.096	.079	.065	.052	.042
1.2	.642	.596	.549	.502	.455	.410	.367	.326	.287	.250	.217	.186	.159	.134	.112	.093	.077	.063	.051	.041
1.3	.626	.580	.534	.488	.443	.398	.357	.316	.279	.243	.211	.181	.154	.130	.109	.091	.075	.061	.049	.039
1.4	.612	.567	.522	.477	.433	.390	.348	.309	.272	.237	.205	.176	.150	.127	.106	.088	.073	.059	.048	.038
1.5	.600	.556	.511	.467	.424	.381	.341	.302	.266	.232	.201	.173	.147	.124	.104	.086	.071	.058	.047	.037
1.6	.590	.546	.502	.459	.416	.375	.335	.297	.261	.228	.197	.169	.144	.122	.102	.084	.069	.057	.046	.037
1.7	.581	.538	.495	.452	.410	.369	.329	.292	.256	.224	.194	.166	.141	.119	.100	.083	.068	.055	.045	.036
1.8	.574	.531	.488	.446	.404	.363	.324	.287	.253	.220	.191	.163	.139	.117	.098	.081	.067	.055	.044	.035
1.9	.567	.525	.482	.440	.399	.359	.320	.283	.249	.217	.188	.161	.137	.116	.097	.080	.066	.054	.043	.035
2.0	.562	.519	.477	.435	.394	.354	.316	.280	.246	.214	.185	.159	.135	.114	.095	.079	.065	.053	.043	.034
2.1	.557	.514	.472	.431	.390	.351	.313	.277	.243	.212	.183	.157	.133	.112	.094	.078	.064	.052	.042	.034
2.2	.552	.510	.468	.427	.386	.347	.310	.274	.241	.210	.181	.155	.132	.111	.093	.077	.063	.051	.042	.033
2.3	.548	.506	.464	.423	.383	.344	.307	.271	.238	.207	.179	.154	.130	.110	.092	.076	.062	.051	.041	.033
2.4	.544	.502	.461	.420	.380	.341	.304	.269	.236	.205	.177	.152	.129	.109	.091	.075	.062	.050	.041	.032
2.5	.540	.499	.458	.417	.377	.338	.302	.267	.234	.204	.176	.151	.128	.108	.090	.075	.061	.050	.040	.032
2.6	.537	.496	.455	.414	.374	.336	.299	.265	.232	.202	.174	.149	.127	.107	.089	.074	.061	.049	.040	.032
2.7	.534	.493	.452	.412	.372	.334	.297	.263	.230	.201	.173	.148	.126	.106	.088	.073	.060	.049	.039	.031
2.8	.532	.490	.450	.409	.370	.332	.296	.261	.229	.199	.172	.147	.125	.106	.088	.073	.060	.048	.039	.031
2.9	.529	.488	.447	.407	.368	.330	.294	.259	.227	.198	.171	.146	.124	.104	.087	.072	.059	.048	.039	.031
3.0	.527	.486	.445	.405	.366	.328	.292	.258	.226	.197	.170	.145	.123	.104	.086	.071	.059	.048	.038	.031
3.1	.525	.484	.443	.403	.364	.326	.290	.256	.225	.195	.169	.144	.122	.103	.086	.071	.058	.047	.038	.030
3.2	.522	.482	.441	.401	.362	.325	.289	.255	.223	.194	.168	.143	.122	.102	.085	.071	.058	.047	.038	.030
3.3	.521	.480	.439	.399	.361	.323	.287	.254	.222	.193	.167	.142	.121	.102	.085	.070	.057	.047	.038	.030
3.4	.519	.478	.438	.398	.359	.322	.286	.253	.221	.192	.166	.142	.120	.101	.084	.070	.057	.046	.037	.030
3.5	.517	.476	.436	.396	.358	.320	.285	.251	.220	.191	.165	.141	.120	.101	.084	.069	.057	.046	.037	.030
3.6	.515	.475	.434	.395	.356	.319	.284	.250	.219	.190	.164	.140	.119	.100	.083	.069	.056	.046	.037	.029
3.7	.514	.473	.433	.393	.355	.318	.283	.249	.218	.190	.163	.140	.118	.100	.083	.069	.056	.046	.037	.029
3.8	.513	.472	.432	.392	.354	.317	.282	.248	.217	.189	.163	.139	.118	.099	.083	.068	.056	.045	.037	.029
3.9	.511	.471	.430	.391	.353	.316	.281	.248	.217	.188	.162	.138	.117	.099	.082	.068	.056	.045	.036	.029

$$C = b_1\sigma_{x1}^2 + b_2\sigma_{x2}^2 = (1)(1) + (1)(1) = 2$$

$$a_1 = \frac{b_1\sigma_{x1}^2}{C} = \frac{(1)(1)}{2} = \frac{1}{2}$$

$$a_2 = \frac{b_2\sigma_{x2}^2}{C} = \frac{(1)(1)}{2} = \frac{1}{2}$$

$$t = \frac{r^2}{C} = \frac{1}{2}$$

From Table C-2, $P_2(a_1, a_2; t) = 0.39347$. This problem can be checked by Table C-3. This table gives corresponding values of P and B where P is the probability under the circular normal distribution for a given value of B, and $B = r/\sigma$ where r is the radius over which the circular normal probability distribution is integrated, and σ is the standard deviation of the circular normal distribution measured along any axis. For the values given above, $B = r/\sigma = 1/1 = 1.000$. From Table C-3,

P	B
0.39	0.9943
0.40	1 0108
0.01	0.0165

By interpolation,

$$P_{B=1.000} = 0.39 + \left(\frac{0.0057}{0.0165}\right)(0.01) = 0.39346$$

This checks the value obtained above.

A second check is possible using Table C-4. This table gives $q(r_d/\sigma, D/\sigma)$ which is the probability that a randomly chosen point will fall outside a circle of radius r_d with center at $(D, 0)$ when the underlying probability distribution is a circular normal distribution centered at the origin and with its two standard deviations equal to σ. Then

$$q\left(\frac{r_d}{\sigma}, \frac{D}{\sigma}\right) = 1 - p\left(\frac{r_d}{\sigma}, \frac{D}{\sigma}\right) \qquad (27)$$

where

$$p\left(\frac{r_d}{\sigma}, \frac{D}{\sigma}\right) = \frac{1}{2\pi\sigma^2}\iint e^{-\frac{1}{2}\left(\frac{x^2+y^2}{\sigma^2}\right)} \tag{28}$$

the area over which this integral is taken,

$$(x - D)^2 + y^2 = r^2 \tag{29}$$

is the equation of a circle of radius r_d centered at $(D,0)$. If D is set equal to zero, $r_d = 1$ and the standard deviation of the circular normal distribution set equal to 1, then $q(r_d/\sigma,\ D/\sigma)$ can be read from Table C-4 and the desired probability found for equation (27). The table of $q(r_d/\sigma,\ D/\sigma)$ uses $(r_d - D)/\sigma$ and D/σ as arguments.

$$\frac{r_d - D}{\sigma} = \frac{(1) - (0)}{1} = 1.000 \qquad \frac{D}{\sigma} = \frac{0}{1} = 0.000$$

From Table C-3, $q(1,0) = 0.607$, so $p(1,0) = 1 - q(1,0) = 1 - 0.607 = 0.393$ which checks the two values previously found.

Problem (2)

Given

$$\alpha = 90^\circ,\ \sigma_1 = \sigma_{x1} = 1,\ \sigma_2 = \sigma_{x2} = 1,\ b_1 = 1,\ b_2 = 1,\ r = 2$$

Find

$$P_2(a_1, a_2; t)$$

This problem uses the same circular probability distribution as was used in problem (1), but the distribution is integrated over a circle whose radius is twice the radius of the circle used in problem (1). Consequently the probability found in this problem should be greater than the probability found in problem (1).

$$C = b_1\sigma_{x1}^2 + b_2\sigma_{x2}^2 = (1)(1) + (1)(1) = 2$$

$$a_1 = \frac{b_1\sigma_{x1}^2}{C} = \frac{(1)(1)}{2} = \frac{1}{2}$$

$$a_2 = \frac{b_2\sigma_{x2}^2}{C} = \frac{(1)(1)}{2} = \frac{1}{2}$$

$$t = \frac{r^2}{C} = \frac{4}{2} = 2$$

From Table C-2, $P_2(a_1, a_2; t) = P_2(0.5, 0.5; 2) = 0.86466$

This answer can be checked by Table C-3. $B = r/\sigma = 2/1 = 2$. From Table C-3,

P	B
0.86	1.9830
0.87	2.0200
0.01	0.0370

By interpolation,

$$P_{B=2} = 0.86 + \frac{0.017}{0.037}(0.01) = 0.8646$$

A second check is possible using Table C-4. From the given quantities,

$$\frac{r_d - D}{\sigma} = \frac{(2) - (0)}{1} = 2.000$$

$$\frac{D}{\sigma} = 0.000$$

From the table

$$q\left(\frac{r_d}{\sigma}, \frac{D}{\sigma}\right) = q(2.000, 0.000) = 0.135$$

and

$$p\left(\frac{r_d}{\sigma}, \frac{D}{\sigma}\right) = 1 - q\left(\frac{r_d}{\sigma}, \frac{D}{\sigma}\right) = 1.000 - 0.135 = 0.865$$

Problem (3)

Given

$$\alpha = 90^\circ,\ \sigma_1 = \sigma_{x1} = 2,\ \sigma_2 = \sigma_{x2} = 2,\ b_1 = 1,\ b_2 = 1,\ r = 2.$$

Find

$$P_2(a_1, a_2; t)$$

This problem is similar to the first problem in that the probability distribution is circular, although in this problem the circular probability distribution is wider and its maximum is lower than the circular distribution in problem (1). Because the area over which the integral is taken is increased in the same proportion as the standard deviation, the probability in this problem is the same as the probability found in problem (1).

$$C = b_1\sigma_{x1}^2 + b_2\sigma_{x2}^2 = (1)(4) + (1)(4) = 8$$

$$a_1 = \frac{b_1\sigma_{x1}^2}{C} = \frac{(1)(4)}{8} = \frac{1}{2}$$

$$a_2 = \frac{b_2\sigma_{x2}^2}{C} = \frac{(1)(4)}{8} = \frac{1}{2}$$

$$t = \frac{r^2}{C} = \frac{4}{8} = \frac{1}{2}$$

From Table C-2, $P_2(a_1, a_2; t) = P_2(0.5; 0.5; 0.5) = 0.39347$. This answer can be checked by Table C-3. $B = r/\sigma = 2/2 = 1$. For $B = 1$, $P = 0.39346$ as shown above for problem (1). A check by means of Table C-4 is also possible.

$$\frac{r_d - D}{\sigma} = \frac{(2) - (0)}{2} = 1$$

$$\frac{D}{\sigma} = 0$$

$$q\left(\frac{r_d}{\sigma}, \frac{D}{\sigma}\right) = q(1, 0) = 0.607$$

$$p\left(\frac{r_d}{\sigma}, \frac{D}{\sigma}\right) = 1 - q\ \frac{r_d}{\sigma}, \frac{D}{\sigma} = 1 - 0.607 = 0.393$$

Problem (4)

Given

$$\alpha = 90^\circ,\ \sigma_1 = \sigma_{x1} = 1,\ \sigma_2 = \sigma_{x2} = 1,\ b_1 = 1,\ b_2 = 2,\ r = 2$$

Find

$$P_2(a_1, a_2; t)$$

This problem uses the same circular probability distribution as was used in problem (1), but the area is now evaluated over an ellipse whose semimajor axis, 2 units long, lies along the x_1 axis and whose semiminor axis, 1 unit long lies along the x_2 axis. Since the same circular probability distribution is used in problems (1), (2) and (4) and the area of integration in problem (4) is intermediate between the areas of integration used in problems (1) and (2), the probability in this problem should fall between the probabilities found in problems (1) and (2).

$$C = b_1\sigma_{x1}^2 + b_2\sigma_{x2}^2 = (1)(1) + (2)(1) = 3$$

$$a_1 = \frac{b_1\sigma_{x1}^2}{C} = \frac{(1)(1)}{3} = \frac{1}{3}$$

$$a_2 = \frac{b_2\sigma_{x2}^2}{C} = \frac{(2)(1)}{3} = \frac{2}{3}$$

$$t = \frac{r^2}{C} = \frac{4}{3}$$

$$P_2(a_1, a_2; t) = P_2\left(\frac{1}{3}, \frac{2}{3}, \frac{4}{3}\right) = 0.74244$$

Table C-3 cannot be used because it is applicable only when the area of integration is circular. Table C-4 can be used to handle integration of a circular normal distribution over an elliptical area. Let $P_r\{b_1x_1^2 + b_2x_2^2 \leqq r^2\}$ denote the probability that a randomly chosen point will lie within an elliptical area $b_1x_1^2 + b_2x_2^2 = r^2$ when x_1 and x_2 are normally distributed with unitary standard deviations, which is the same as the integral of a circular normal distribution with the same standard deviation integrated over the area defined by that ellipse. Then

$$P_r\{b_1x_1^2 + b_2x_2^2 \leqq r^2\} = q(B, A) - q(A, B)$$

where

$$A = \frac{1}{2}\left[\left(\frac{r^2}{b_1}\right)^{\frac{1}{2}} + \left(\frac{r^2}{b_2}\right)^{\frac{1}{2}}\right] = \frac{1}{2}\left[\left(\frac{4}{1}\right)^{\frac{1}{2}} + \left(\frac{4}{2}\right)^{\frac{1}{2}}\right] = 1.707$$

and

$$B = \frac{1}{2}\left|\left(\frac{r^2}{b_1}\right)^{\frac{1}{2}} - \left(\frac{r^2}{b_2}\right)^{\frac{1}{2}}\right| = \frac{1}{2}\left|\left(\frac{4}{1}\right)^{\frac{1}{2}} - \left(\frac{4}{2}\right)^{\frac{1}{2}}\right| = 0.293$$

$$q(B,\ A) = q\left(\frac{r_d}{\sigma},\ \frac{D}{\sigma}\right) \quad .$$

In order to find $q(B,\ A)$ Table C-4 must be entered using the values $(r_d - D/\sigma$ and D/σ for the arguments.

$$q(B,\ A) = q\left(\frac{r_{d1}}{\sigma},\ \frac{D_1}{\sigma}\right)$$

$$\frac{r_{d1} - D_1}{\sigma} = 0.293 - 1.707 = -1.414$$

$$\frac{D_1}{\sigma} = 1.707$$

By interpolation,

$$q(B,\ A) = 0.990$$

$$q(A,\ B) = q\left(\frac{r_{d2}}{\sigma},\ \frac{D_2}{\sigma}\right) = q(1.707,\ 0.293)$$

$$\frac{r_{d2} - D_2}{\sigma} = 1.707 - 0.293 = 1.414$$

$$\frac{D_2}{\sigma} = 0.293$$

By interpolation,

$$q(A,\ B) = 0.248 \quad .$$

Finally

$$P_r\{b_1x_1^2 + b_2x_2^2 \leq r^2\} = q(B, A) - q(A, B)$$

$$= 0.990 - 0.248 = 0.742$$

This checks the previously obtained value of $P_2(a_1, a_2; t)$

Problem (5)

Given

$$\alpha = 90°, \ \sigma_1 = \sigma_{x1} = 1, \ \sigma_2 = \sigma_{x2} = 2, \ b_1 = 1, \ b_2 = 1, \ r = 1$$

Find $P_2(a_1, a_2; t)$

Because the standard deviations of the bivariate normal probability distribution are unequal, the contour lines of the distribution are ellipses with their major axes lying along the x_2 axis. The integral of the elliptical probability distribution will be found over a circle whose radius is one.

$$C = b_1\sigma_{x1}^2 + b_2\sigma_{x2}^2 = (1)(1) + (1)(4) = 5$$

$$a_1 = \frac{b_1\sigma_{x1}^2}{C} = \frac{(1)(1)}{5} = 0.2$$

$$a_2 = \frac{b_2\sigma_{x2}^2}{C} = \frac{(1)(4)}{5} = 0.8$$

$$t = \frac{r^2}{C} = \frac{1}{5} = 0.2$$

$$P_2(a_1, a_2; t) = P_2(0.2, 0.8; 0.2) = 0.21529$$

Problem (6)

Given

$$\alpha = 90°, \ \sigma_1 = \sigma_{x1} = 1, \ \sigma_2 = \sigma_{x2} = 2, \ b_1 = 1, b_2 = 1, \ r = 2$$

Find

$$P_2(a_1, a_2; t)$$

This is the same as the preceding problem, except that the radius of the circle over which the elliptical distribution is to be integrated has been doubled.

$$C = b_1\sigma^2_{x1} + b_2\sigma^2_{x2} = (1)(1) + (1)(4) = 5$$

$$a_1 = \frac{b_1\sigma^2_{x1}}{C} = 0.2$$

$$a_2 = \frac{b_2\sigma^2_{x2}}{C} = 0.8$$

$$t = \frac{r^2}{C} = \frac{(2)^2}{5} = 0.8$$

$$P_2(a_1, a_2; t) = P_2(0.2, 0.8; 0.8) = 0.59009$$

Problem (7)

Given

$$\alpha = 90°,\ \sigma_1 = \sigma_{x1} = 1,\ \sigma_2 = \sigma_{x2} = 2,\ b_1 = 4,\ b_2 = 1,\ r = 2$$

Find

$$P_2(a_1, a_2; t)$$

The elliptical distribution is the same as in the preceding two problems, but now the integral of the elliptical distribution is to be integrated over an ellipse whose major axis is parallel to the major axes of the ellipses of equal probability density on the elliptical distribution.

$$C = b_1\sigma^2_{x1} + b_2\sigma^2_{x2} = (4)(1) + (1)(4) = 8$$

$$a_1 = \frac{b_1\sigma^2_{x1}}{C} = \frac{(4)(1)}{8} = 0.5$$

$$a_2 = \frac{b_2\sigma^2_{x2}}{C} = \frac{(1)(4)}{8} = 0.5$$

$$t = \frac{r^2}{C} = \frac{4}{8} = 0.5$$

$$P_2(a_1, a_2; t) = P_2(0.5, 0.5; 0.5) = 0.39347$$

Problem (8)

Given

$\alpha = 90°$, $\sigma_1 = \sigma_{x2} = 1$, $\sigma_2 = \sigma_{x2} = 2$, $b_1 = 1$, $b_2 = 4$, $r = 2$

Find

$P_2(a_1, a_2; t)$

This problem uses the same elliptical probability distribution and the same elliptical area over which the elliptical distribution is to be integrated, except that in this problem the elliptical area has been turned 90° so that its major axis is parallel to the minor axes of the contours of equal probability density of the elliptical probability distribution.

$$C = b_1\sigma_{x1}^2 + b_2\sigma_{x2}^2 = (1)(1) + (4)(4) = 17$$

$$a_1 = \frac{b_1\sigma_{x1}^2}{C} = \frac{(1)(1)}{17} = 0.05882$$

$$a_2 = \frac{b_2\sigma_{x2}^2}{C} = \frac{(4)(4)}{17} = \frac{16}{17} = 0.9412$$

$$t = \frac{r^2}{C} = \frac{4}{17} = 0.2353$$

$$P_2(a_1, a_2; t) = P_2(0.05882, 0.9412; 0.2353) = 0.32623$$

Problem (9)

Given

$\alpha = 60°$, $\sigma_1 = 1$, $\sigma_2 = 1$, $b_1 = 1$, $b_2 = 1$, $r = 1$

Find

$P_2(a_1, a_2; t)$

In all of the previous problems the lines of position have been assumed to intersect at right angles. In this and the remaining problems in this memorandum, angles of intersection other than 90° will be used. The area over which the elliptical probability distribution is to be integrated is a circle of unitary radius centered on the origin, so equations (24) to (26) can be used.

$$a_1 = \frac{1}{2}\left\{1 + \left[1 - \frac{4(\sin^2 \alpha)\sigma_1^2\sigma_2^2}{(\sigma_1^2 + \sigma_1^2)^2}\right]^{1/2}\right\}$$

$$= \frac{1}{2}\left\{1 + \left[1 - \frac{4\left(\frac{3}{4}\right)(1)(1)}{(1 + 1)^2}\right]^{1/2}\right\} = \frac{1}{2}\left\{1 + \left[\frac{1}{4}\right]^{1/2}\right\} = \frac{3}{4}$$

$$a_2 = \frac{1}{2}\left\{1 - \left[1 - \frac{4(\sin^2 \alpha)\sigma_1^2\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2}\right]^{1/2}\right\} = \frac{1}{2}\left\{1 - \left[\frac{1}{4}\right]^{1/2}\right\} = \frac{1}{4}$$

$$t = \frac{r^2 \sin^2 \alpha}{\sigma_1^2 + \sigma_2^2} = \frac{(1)\left(\frac{3}{4}\right)}{1 + 1} = \frac{3}{8}$$

$$P_2(a_1, a_2; t) = P_2(0.75, 0.25; 0.375) = 0.34230$$

It is also possible to solve this problem by first solving for σ_{x1}^2 and σ_{x2}^2 and then using equations (14) to (17), as in the previous problems.

$$\sigma_{x1}^2 = \frac{1}{2 \sin^2 \alpha}\{\sigma_1^2 + \sigma_2^2 + [(\sigma_1^2 + \sigma_2^2)^2 - 4(\sin^2 \alpha)\sigma_1^2\sigma_2^2]^{1/2}\}$$

$$= \frac{1}{2\left(\frac{3}{4}\right)}\left\{1 + 1 + \left[(1 + 1)^2 - 4\left(\frac{3}{4}\right)(1)(1)\right]^{1/2}\right\} = 2$$

$$\sigma_{x2}^2 = \frac{1}{2 \sin^2 \alpha}\{\sigma_1^2 + \sigma_2^2 - [(\sigma_1^2 + \sigma_2^2)^2 - 4(\sin^2 \alpha)\sigma_1^2\sigma_2^2]^{1/2}\} = \frac{2}{3}$$

From these values, the contours of equal probability density are ellipses satisfying the equation $(x_1^2/2) + (3x_2^2/2) = k_n^2$. The ellipse with $k_n = 1$

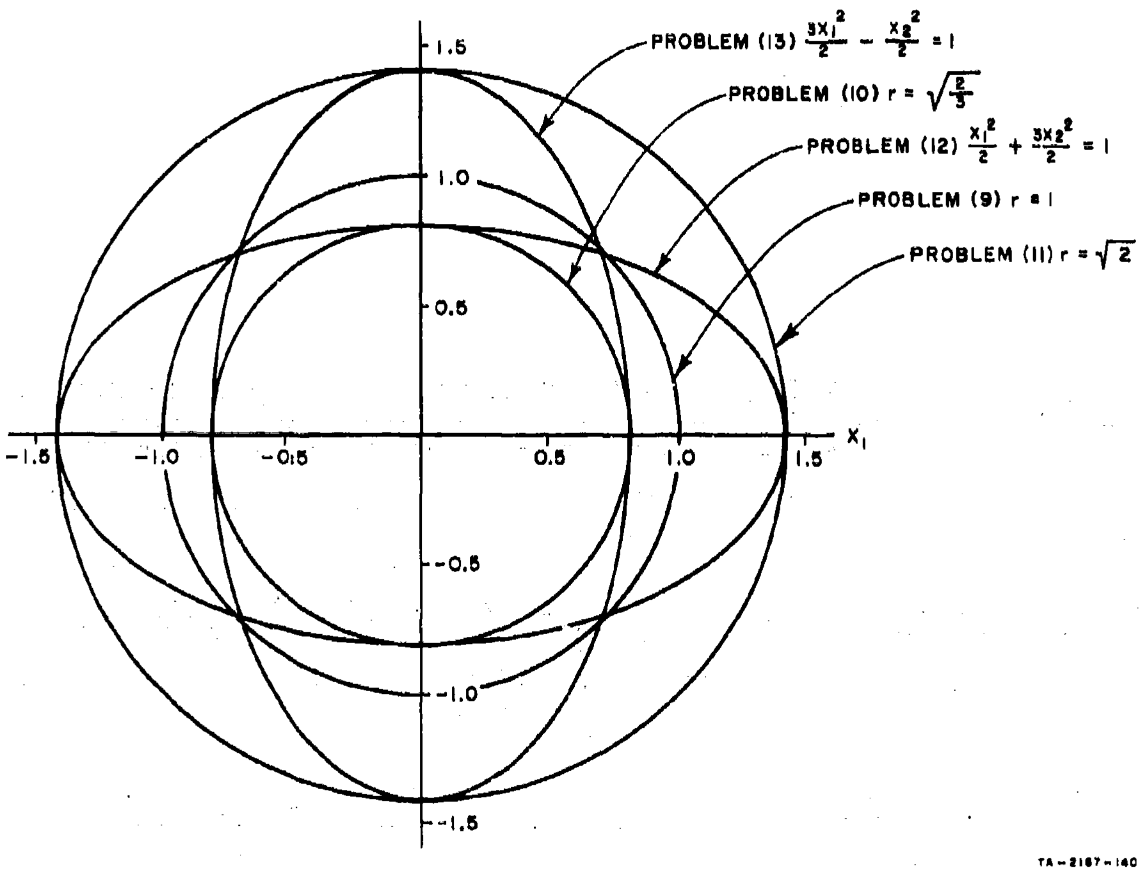

FIG. C-1 PROBLEM ILLUSTRATIONS

is shown in Fig. C-1 together with the circle over which the elliptical distribution is integrated.

$$C = b_1\sigma_{x1}^2 + b_2\sigma_{x2}^2 = 2 + \frac{2}{3} = \frac{8}{3}$$

$$a_1 = \frac{b_1\sigma_{x1}^2}{C} = \frac{(1)(2)}{\frac{8}{3}} = 0.75$$

$$a_2 = \frac{b_2\sigma_{x2}^2}{C} = \frac{(1)\left(\frac{2}{3}\right)}{\frac{8}{3}} = 0.25$$

$$t = \frac{r^2}{C} \quad \frac{1}{\frac{8}{3}} = 0.375$$

Problem (10)

Given

$$\alpha = 60^\circ,\ \sigma_1 = 1,\ \sigma_2 = 1,\ b_1 = 1,\ b_2 = 1,\ r = \sqrt{\frac{2}{3}}$$

Find

$$P_2(a_1, a_2; t)$$

The elliptical probability distribution in this problem is the same as in the preceding problem, and the only difference is that the radius of the circle over which the elliptical probability distribution is to be integrated is reduced to 0.8165. This circle is also shown in Fig. C-1. Therefore a_1 and a_2 are the same as in the preceding problem.

$$a_1 = 0.75$$

$$a_2 = 0.25$$

$$t = \frac{r^2 \sin^2 \alpha}{\sigma_1^2 + \sigma_2^2} = \frac{\left(\frac{2}{3}\right)\left(\frac{3}{4}\right)}{1 + 1} = 0.25$$

$$P_2(a_1, a_2; t) = P_2(0.75, 0.25; 0.25) = 0.24601$$

Problem (11)

Given

$$\alpha = 60^\circ,\ \sigma_1 = 1,\ \sigma_2 = 1,\ b_1 = 1,\ b_2 = 1,\ r = \sqrt{2}$$

Find

$$P_2(a_1, a_2; t)$$

This problem is the same as the preceding two, except that the radius of the area over which the elliptical probability distribution is to be integrated is now 1.414. This circle is also shown in Fig. C-1.

$$a_1 = 0.75$$

$$a_2 = 0.25$$

$$t = \frac{r^2 \sin^2 \alpha}{\sigma_1^2 + \sigma_2^2} = \frac{2\left(\frac{3}{4}\right)}{2} = 0.75$$

$$P_2(a_1, a_2; t) = P_2(0.75, 0.25; 0.75) = 0.55620$$

Problem (12)

Given

$$\alpha = 60°,\ \sigma_1 = 1,\ \sigma_2 = 1,\ b_1 = \frac{1}{2},\ b_2 = \frac{3}{2},\ r = 1.$$

Find

$$P_2(a_1, a_2; t)$$

This problem uses the same elliptical probability distribution as the three preceding problems, but now the distribution is to be integrated over an area given by the ellipse $(x_1^2/2) + (3x_2^2/2) = 1$. This is the same ellipse as the contour line in Fig. C-1, where $k_n = 1$. From problem 9, $\sigma_{x1}^2 = 2$ and $\sigma_{x2}^2 = 2/3$.

$$C = b_1\sigma_{x1}^2 + b_2\sigma_{x2}^2 = \frac{1}{2}(2) + \left(\frac{3}{2}\right)\left(\frac{2}{3}\right) = 2$$

$$a_1 = \frac{b_1\sigma_{x1}^2}{C} = \frac{\frac{1}{2}(2)}{2} = 0.5$$

$$a_2 = \frac{b_2\sigma_{x2}^2}{C} = 0.5$$

$$t = \frac{r^2}{C} = 0.5$$

$$P_2(a_1, a_2; t) = P_2(0.5, 0.5; 0.5) = 0.39347$$

Problem (13)

Given

$$\alpha = 60°,\ \sigma_1 = 1,\ \sigma_2 = 1,\ b_1 = \frac{3}{2},\ b_2 = \frac{1}{2},\ r = 1$$

Find

$$P_2(a_1, a_2; t)$$

This problem uses the same elliptical probability distribution and an elliptical area over which the elliptical probability is to be integrated

that is the same size and shape as the elliptical area in the preceding problem, except that for this problem the elliptical area has been rotated through 90°. The relationship of this ellipse to the probability distribution can be seen in Fig. C-1.

$$C = b_1\sigma_{x1}^2 + b_2\sigma_{x2}^2 = (2)(3/2) + (2/3)(1/2) = 3(1/3)$$

$$a_1 = \frac{b_1\sigma_{x1}^2}{C} = \frac{(2)(3/2)}{3(1/3)} = 0.9$$

$$a_2 = \frac{b_2\sigma_{x2}^2}{C} = \frac{(2/3)(1/2)}{3(1/3)} = 0.1$$

$$t = \frac{r^2}{C} = \frac{1}{3(1/3)} = 0.3$$

$$P_2(a_1, a_2; t) = P_2(0.9, 0.1; 0.3) = 0.34945$$

Problem (14)

Given two lines of position which intersect at an angle of 65° and which have standard deviations of 700 ft and 900 ft, respectively. Find the probability that the indicated position is within a quarter of a nautical mile of the true position.

By the given quantities, $\alpha = 65°$, $\sigma_1 = 700$ ft., $\sigma_2 = 900$ ft., $b_1 = 1$, $b_2 = 1$, $r = 1520$ ft.

$$a_1 = \frac{1}{2}\left\{1 + \left[1 - \frac{4(\sin^2 \alpha)\sigma_1^2\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2}\right]^{1/2}\right\}$$

$$= \frac{1}{2}\left\{1 + \left[1 - \frac{4(0.82139)(700)^2(900)^2}{(700^2 + 900^2)^2}\right]^{1/2}\right\} = 0.739$$

$$a_2 = 0.261$$

$$t = \frac{r^2}{\sigma_1^2 + \sigma_2^2} = \frac{(1520)^2(0.82139)}{(700)^2 + (900)^2} = 1.460$$

$$P_2(a_1, a_2; t) = P_2(0.739, (.261; 1.460) = 0.77624$$

Problem (15)

Given the same conditions in the preceding problem, but the probability that the true position is within half a nautical mile is desired.

$$a_1 = 0.739$$

$$a_2 = 0.261$$

$$t = \frac{r^2 \sin^2 \alpha}{\sigma_1^2 + \sigma_2^2} = \frac{(3040)^2(0.82139)}{(700)^2 + (900)^2} = 5.84$$

$$P_2(a_1, a_2; t) = P_2(0.739, 0.261; 5.84) = 0.99404$$

APPENDIX D

DERIVATION OF METHOD 1 FORMULAS

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DERIVATION OF METHOD 1 FORMULAS

1. INTRODUCTION

Method 1, as described in the body of the memorandum, initially treats the special case where $\sigma_1 = \sigma_2$. Early in the performance of this project the probability curves for this case shown in Fig. 7, Section 2, were calculated by a digital computer program. However, because of their limited application, little use could be made of them until the further operations involving the fictitious σ^* and α^* were developed. A step in the development of these functions was the derivation of an alternative set of formulas for σ_x and σ_y. This alternative set involves the computation of auxiliary functions which were then found to lead to the formulas for σ^* and α^*.

This appendix discusses in order the development of the probability curves $P(R/\sigma, \alpha)$; the development of the alternative formulas for σ_x and σ_y; and then gives a derivation for the formulas of Figs. 9 and 10 of Section 2 that utilizes relationships developed in Appendix B as starting points.

2. DEVELOPMENT OF $P(R/\sigma, \alpha)$ CURVES

Given, as in Appendix B, the intersection of two lines of position at an angle α. However, this derivation is limited to the special case where the two associated standard deviations, σ_1 and σ_2 are equal. In this special case, Equation (5) of Appendix B may be simplified to

$$P = \frac{1}{2\pi\sigma^2} \iint e^{-(1/2\sigma^2)(u^2+v^2)}\, du dv \quad . \tag{1}$$

Since we want to work in orthogonal coordinates we shall proceed to transform to new orthogonal coordinates ξ and η where the η axis bisects the angle α (Fig. D-1). The choice of this position is to reduce the algebra later on.

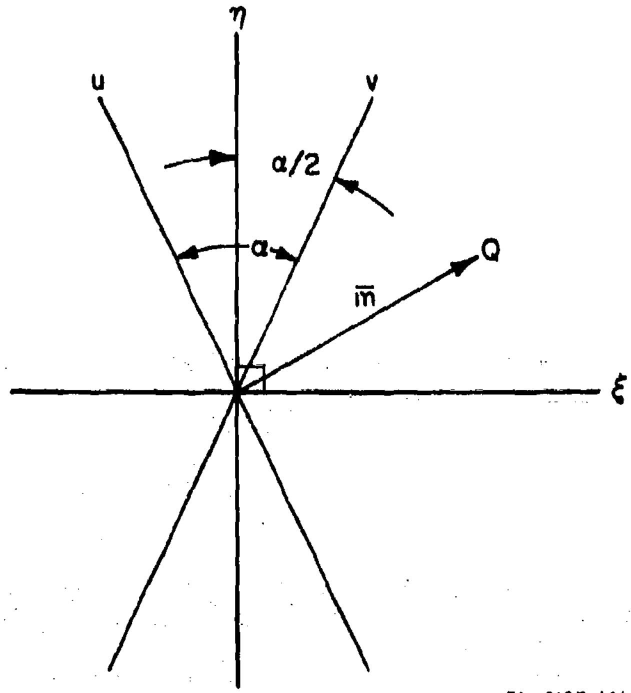

FIG. D-1 TRANSFORMATION OF AXES

Take an arbitrary point Q. We want to find its position along the x and y axis in terms of ξ and η. Take the vector $\overline{m}$ from the origin 0 to Q. $\overline{m} = \xi\, \overline{l}_\xi + \eta \overline{l}_\eta$ where $\overline{l}_\xi$ and $\overline{l}_\eta$ are unit vectors along the ξ and η axes respectively.

$\overline{l}_x$ and $\overline{l}_y$ are unit vectors along the u and v axes respectively. To find the projections of $\overline{m}$ on the u and v axes, we dot it with the respective unit vectors.

$$u = \overline{m} \cdot \overline{l}_x = \xi \overline{l}_\xi \cdot \overline{l}_x + \eta \overline{l}_\eta \cdot \overline{l}_x = \xi \sin\frac{\alpha}{2} + \eta \cos\frac{\alpha}{2} \tag{2}$$

$$v = \overline{m} \cdot \overline{l}_y = \xi \overline{l}_\xi \cdot \overline{l}_y + \eta \overline{l}_\eta \cdot \overline{l}_y = - \sin\frac{\alpha}{2} + \eta \cos\frac{\alpha}{2} \tag{3}$$

$$\therefore u^2 + v^2 = 2\left(\xi^2 \sin^2\frac{\alpha}{2} + \eta^2 \cos^2\frac{\alpha}{2}\right) \tag{4}$$

$$dudv = J\begin{pmatrix} u & v \\ \xi & \eta \end{pmatrix} d\xi d\eta = \sin \alpha d\xi d\eta \tag{5}$$

$$dp = \frac{1}{2\pi\sigma^2} e^{-\left[\frac{\xi^2 \sin^2 \frac{\alpha}{2} + \eta^2 \cos^2 \frac{\alpha}{2}}{\sigma^2}\right]} \sin \alpha d\xi d\eta \quad . \tag{6}$$

We are now in regular orthogonal axes ξ and η. To find the probabilities in the circle of radius R we integrate dp over the circle.

$$p = 4 \int_{\xi=0}^{\xi=R} \int_{\eta=0}^{\eta=\sqrt{R^2-\xi^2}} \frac{1}{2\pi\sigma^2} e^{-\left[\frac{\xi^2 \sin^2 \frac{\alpha}{2} + \eta^2 \cos^2 \frac{\alpha}{2}}{\sigma^2}\right]} \sin \alpha d\xi d\eta \quad . \tag{7}$$

let

$$\xi = r \cos \alpha \sin \frac{\alpha}{2} \tag{8}$$

and

$$\eta = r \sin \alpha \cos \frac{\alpha}{2} \tag{9}$$

$$d\xi d\eta = J\begin{pmatrix} \theta & \eta \\ r & \phi \end{pmatrix} dr d\phi = \frac{2r}{\sin \theta} dr d\phi \tag{10}$$

and

$$r = R/\sqrt{\frac{\cos^2 \phi}{\sin^2 \frac{\alpha}{2}} + \frac{\sin^2 \phi}{\sin^2 \frac{\alpha}{2}}} \quad . \tag{11}$$

We therefore have

$$p = 4 \int_{\phi=0}^{\phi=\pi/2} \int_{r=0}^{r=R/\sqrt{\frac{\cos^2 \phi}{\sin^2 \frac{\alpha}{2}} + \frac{\sin^2 \phi}{\cos^2 \frac{\alpha}{2}}}} \frac{1}{2\pi\sigma^2} e^{-\frac{r^2}{\sigma^2}} 2r dr d\phi \quad . \tag{12}$$

Performing the first integration

$$p = \left[-\frac{2}{\pi} \int_0^{\pi/2} e^{-\frac{r^2}{2}} \right]_0^{R \Big/ \sqrt{\frac{\cos^2 \phi}{\sin^2 \frac{\alpha}{2}} + \frac{\sin^2 \phi}{\cos^2 \frac{\alpha}{2}}}} d\phi \tag{13}$$

$$p = \left[\frac{2}{\pi} \int_0^{\pi/2} 1 - e^{-\frac{R^2}{\sigma^2} \Big/ \left(\frac{\cos^2 \phi}{\sin^2 \frac{\alpha}{2}} + \frac{\sin^2 \phi}{\cos^2 \frac{\alpha}{2}} \right)^{\alpha}} \right] d\phi \tag{14}$$

or

$$p = 1 - \frac{2}{\pi} \int_0^{\pi/2} e^{-\frac{R^2}{\sigma^2} \Big/ \left(\frac{\cos^2 \phi}{\sin^2 \frac{\phi}{2}} + \frac{\sin^2 \phi}{\cos^2 \frac{\phi}{2}} \right)^{\alpha}} d\phi \tag{15}$$

This integral cannot be evaluated analytically. A solution was obtained by a digital computer program and the results have been presented in Fig. 7, Section 2, and numerically in Table I, Section 2.

3. ALTERNATIVE FORMULAS FOR σ_x AND σ_y

Additional formulas for σ_x and σ_y utilizing auxiliary functions were also developed in the course of the analytical investigations of this study. The additional formulas are given below, followed by the derivation that is developed using the methods and notation of vector algebra.

$$\sigma_x = \frac{\sigma}{\sqrt{2}} \frac{\sin 2\beta}{\sqrt{1 - \sqrt{1 - \sin^2 2\beta \sin^2 \alpha}}} \tag{16}$$

$$\sigma_y = \frac{\sigma}{\sqrt{2}} \frac{\sin 2\beta}{\sqrt{1 + \sqrt{1 - \sin^2 2\beta \sin^2 \alpha}}} \tag{17}$$

where

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$$

$$\beta = \arctan \sigma_2/\sigma_1$$

from which

$$\sin 2\beta = \frac{2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2}$$

The coordinate systems are exhibited in Fig. D-2.

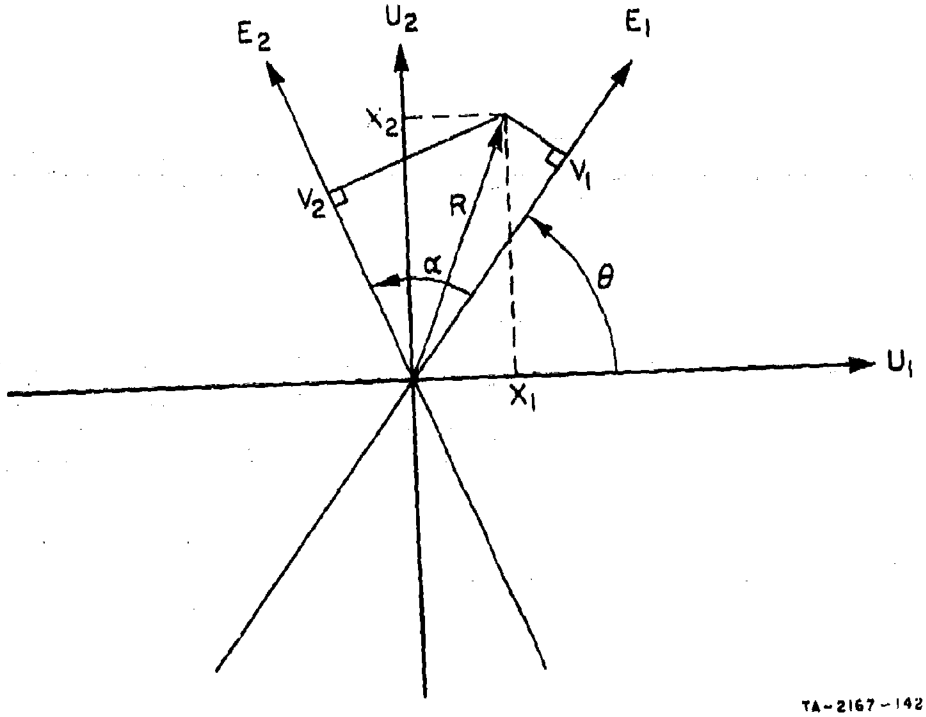

FIG. D-2 COORDINATE SYSTEM

Since

$$V_1 = R \cdot E_1$$
$$V_2 = R \cdot E_2$$

and

$$R = x_1 U_1 + x_2 U_2 \;:$$

$$\begin{cases} V_1 = x_1 \cos\theta + x_2 \sin\theta \\ V_2 = x_1 \cos(\alpha + \theta) + x_2 \sin(\alpha + \theta) \end{cases} \tag{18}$$

Now if (A,B) are a pair of vectors where $A = (A_1,A_2) B = (B_1,B_2)$ note the identity:

$$(A_1x_1 + B_1x_2)^2 + (A_2x_1 + B_2x_2)^2 = |A|^2 x_1^2 + 2(A \cdot B)x_1x_2 + |B|^2 x_2^2$$
$$= |x_1A + x_2B|^2 \quad . \qquad (19)$$

Let

$$\begin{aligned} A &= [\cos \theta/\sigma_g, \quad \cos (\alpha + \theta)/\sigma_h] \\ B &= [\sin \theta/\sigma_g, \quad \sin (\alpha + \theta)/\sigma_h] \end{aligned} \quad . \qquad (20)$$

Choose θ so that $A \cdot B = 0$, that is,

$$\frac{\cos \theta \sin \theta}{\sigma_g^2} + \frac{\cos (\alpha + \theta) \sin (\alpha + \theta)}{\sigma_h^2} = 0$$

or equivalently:

$$\theta = \frac{1}{2} \left[\arctan \left(\frac{\sigma_g^2 - \sigma_h^2}{\sigma_g^2 + \sigma_h^2} \tan \alpha \right) - \alpha \right] \quad . \qquad (21)$$

By this choice of θ with the help of (20):

$$|A|^2 |B|^2 = |A \times B|^2 = \left(\frac{\sin \alpha}{\sigma_g \sigma_h} \right)^2 \quad . \qquad (22)$$

Define

$$\begin{cases} \sigma_g = \sigma \cos \beta \\ \sigma_h = \sigma \sin \beta \end{cases} \quad . \qquad (23)$$

Then (21) may be expressed

$$\theta = \frac{1}{2} \{\arctan [\cos (2\beta) \tan \alpha] - \alpha\} \quad . \qquad (24)$$

Temporarily define

$$G = \frac{2}{\sigma^2 \sin^2 2\beta}$$

$$H = \cos 2\theta \sin^2 \beta + \cos (2\alpha + 2\theta) \cos^2 \beta$$

then

$$|A|^2 = G + H \text{ and } |B|^2 = G - H \quad .$$

Now in principle one could use (24) to eliminate θ from (21). We shall take a computationally simpler alternative. With the aid of (22) we write:

$$G^2 - H^2 = |A|^2|B|^2 = \frac{\sin^2\alpha}{\sigma_1^2\sigma_2^2} = \frac{4\sin^2\alpha}{\sigma^4\sin^2 2\beta} = 2G\sin^2\alpha/\sigma^2$$

Hence

$$H^2 = G^2 - 2G\sin^2\alpha/\sigma^2$$

and

$$|A|^2, |B|^2 = G \pm (G^2 - 2G\sin^2\alpha/\sigma^2)^{1/2} \quad .$$

Therefore

$$\frac{1}{|A|^2}, \frac{1}{|B|^2} = \frac{\sigma^2}{2G\sin^2\alpha}\left[G \pm (G^2 - 2G\sin^2\alpha/\sigma^2)^{1/2}\right]$$

$$\frac{\sigma^2\csc^2\alpha}{2}\left[1 \pm \left(1 - \frac{2\sin^2\alpha}{G/\sigma^2}\right)^{1/2}\right]$$

$$= \frac{\sigma^2\csc^2\alpha}{2}\left[1 \pm (1 - \sin^2 2\beta\sin^2\alpha)^{1/2}\right]$$

$$= \frac{\frac{1}{2}\sigma^2\sin^2 2\beta}{1 \pm (1 - \sin^2 2\beta\sin^2\alpha)^{1/2}} \quad .$$

By letting $\alpha = \pi/2$ we identify in the foregoing "+" with $1/|B|^2$ and "−" with $1/|A|^2$. Finally we note that by (19):

$$1/|A|^2 = \sigma_x$$

$$1/|B|^2 = \sigma_y \quad .$$

The auxiliary functions σ and β are useful in the formulas for σ^* and α^* that form the basis for the Figs. 8 and 9 of Section 2. The formulas for these special functions were originally derived from the just derived

formulas for σ_x and σ_y. A subsequent derivation that starts from formulas of Appendix B, however, is easier to follow.

Formulas (16) and (17) of this section may be shown to be equivalent to the equations developed in Appendix B (36) and (37). To do so it is necessary to substitute the equivalents for σ and β given in this expression.

The first step is to square Equation (16) to eliminate one radial in the denominator as a simplification

$$\sigma_x^2 = \frac{\sigma^2}{2} \frac{\sin^2 2\beta}{1 - \sqrt{1 - \sin^2 2\beta \sin^2 \alpha}}$$

Now, substituting

$$\sigma^2 = \sigma_1^2 + \sigma_2^2$$

$$\sin^2 2\beta = \frac{4\sigma_1^2\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2}$$

$$\sigma_f^2 = \left[\frac{2\sigma_1^2\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)}\right] \frac{1}{\left[1 - \sqrt{1 - \frac{4\sigma_1^2\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2} \sin^2 \alpha}\right]}$$

$$= \frac{2\sigma_1^2\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)} \times \frac{1 + \sqrt{1 - \frac{4\sigma_1^2\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2} \sin^2 \alpha}}{1 - \left[1 - \frac{4\sigma_1^2\sigma_2^2}{(\sigma_1^2 + \sigma_2^2)^2} \sin^2 \alpha\right]}$$

$$= (\sigma_1^2 + \sigma_2^2) \times \frac{1 + \frac{1}{\sigma_1^2 + \sigma_2^2} \sqrt{(\sigma_1^2 + \sigma_2^2) - 4 \sin^2 \alpha\sigma_1^2\sigma_2^2}}{2(\sigma_1^2 + \sigma_2^2)^2 \sin^2 \alpha}$$

$$\sigma_f^2 = (\sigma_1^2 + \sigma_2^2) \times \frac{\left[1 + \dfrac{1}{\sigma_1^2 + \sigma_2^2}\sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4\sin^2 \alpha\sigma_1^2\sigma_2^2}\right]}{2\sin^2 \alpha}$$

$$= \frac{(\sigma_1^2 + \sigma_2^2) + \sqrt{(\sigma_1^2 + \sigma_2^2)^2 - 4\sin^2 \sigma_1^2\sigma_2^2}}{2\sin^2 \alpha}$$

QED

Similarly, except for changes in sign we may show that σ_f^2 from this section is equivalent to Equation (37) of Appendix B.

4. DERIVATION OF FORMULAS USED FOR CURVE AND NOMOGRAMS IN SECTION 2

In the body of the memorandum, Section 2, these formulas were given as the basis for Figs. 8 and 9. This curve and nomogram permits the conversion of unequal σ_1 and σ_2 associated with the intersection angle α to a fictitious pair of equal standard deviations σ^* and a fictitious intersection angle α^*. Following these conversions, the probability curves of Fig. 7, Section 2, may be entered.

$$\alpha^* = \arcsin (\sin 2\beta \sin \alpha)$$

$$R/\sigma^* = \frac{R\sqrt{2}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \csc 2\beta \quad .$$

The derivation starts from Equations (34) and (35) of Appendix B.

In any error ellipse we have

$$\sigma_x^2\sigma_y^2 = \frac{\sigma_1^2\sigma_2^2}{\sin^2 \alpha} \qquad \text{Appendix B (34)} \qquad (27)$$

and

$$\sigma_x^2 + \sigma_y^2 = \frac{\sigma_1^2 + \sigma_2^2}{\sin^2 \alpha} \qquad \text{Appendix B (35)} \qquad (28)$$

We assume that the same distribution may also be described by fictitious functions σ^*, σ^*, and α^*. Thus substituting in the formulas above

$$\sigma_x^2\sigma_y^2 = \frac{\sigma^{*4}}{\sin^2 \alpha^*} \tag{29}$$

and

$$\sigma_x^2 + \sigma_y^2 = \frac{2\sigma^{*2}}{\sin^2 \alpha^*} \tag{30}$$

equating the right-hand sides of Equations (27) with (29) and (28) with (30), we obtain

$$\frac{\sigma_1^2\sigma_2^2}{\sin^2 \alpha} = \frac{\sigma^{*4}}{\sin^2 \alpha^*} \tag{31}$$

and

$$\frac{\sigma_1^2 + \sigma_2^2}{\sin^2 \alpha} = \frac{2\sigma^{*2}}{\sin^2 \alpha^*} . \tag{32}$$

From the second of these we get

$$\sigma^{*2} = \frac{\sin^2 \alpha^*(\sigma_1^2 + \sigma_2^2)}{2 \sin^2 \alpha} . \tag{33}$$

Substituting in the first we get

$$\frac{\sigma_1^2\sigma_2^2}{\sin^2 \alpha} = \frac{\sin^4 \alpha^*(\sigma_1^2 + \sigma_2^2)^2}{4 \sin^4 \alpha \sin^2 \alpha^*} \tag{34}$$

$$\sin^2 \alpha^* = \frac{4 \sin^4 \alpha\sigma_1^2\sigma_2^2}{\sin^2 \alpha(\sigma_1^2 + \sigma_2^2)^2} \tag{35}$$

$$\sin \alpha^* = \frac{2\sigma_1\sigma_2 \sin \alpha}{\sigma_1^2 + \sigma_2^2} . \tag{36}$$

From Section 3 of this appendix we obtain the following:

Define

$$\beta \equiv \arctan \sigma_2/\sigma_1 \tag{37}$$

Then

$$\sin \beta = \frac{\sigma_2}{\sqrt{\sigma_1^2 + \sigma_2^2}} \tag{38}$$

and

$$\cos \beta = \frac{\sigma_1}{\sqrt{\sigma_1^2 + \sigma_2^2}} \tag{39}$$

Thus

$$\sin 2\beta = \frac{2\sigma_1\sigma_2}{\sigma_1^2 + \sigma_2^2} \tag{40}$$

Substituting (40) in (36) we get

$$\sin \alpha^* = \sin 2\beta \sin \alpha \tag{41}$$

$$\alpha^* = \arcsin (\sin 2\beta \sin \alpha) \tag{42}$$

Substituting Equation (41) in the expression for σ^{*2} (33)

$$\sigma^{*2} = \frac{\sin^2 2\beta \sin^2 \alpha(\sigma_1^2 + \sigma_2^2)}{2 \sin^2 \alpha} \tag{43}$$

$$\sigma^* = \frac{\sin 2\beta\sqrt{\sigma_1^2 + \sigma_2^2}}{\sqrt{2}} \tag{44}$$

$$\frac{1}{\sigma^*} = \frac{\sqrt{2}}{\sin 2\beta\sqrt{\sigma_1^2 + \sigma_2^2}} \tag{45}$$

$$= \frac{\sqrt{2} \csc 2\beta}{\sqrt{\sigma_1^2 + \sigma_2^2}} \tag{46}$$

Thus

$$R/\sigma^* = \frac{R\sqrt{2} \csc 2\beta}{\sqrt{\sigma_1^2 + \sigma_2^2}} \tag{47}$$

where

$$\csc 2\beta = \frac{1}{\sin 2\beta} = \frac{\sigma_1^2 + \sigma_2^2}{2\sigma_1\sigma_2} . \tag{48}$$

APPENDIX E

COMBINATION OF MULTIPLE ELLIPSES

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APPENDIX E

COMBINATION OF MULTIPLE ELLIPSES

The basic elements of the combination of several ellipses with random orientations of their axes have been given in Section 3 of the basic Memorandum (see Fig. 15). Each ellipse is expressed by its individual values of σ_x and σ_y and the orientation angle θ with respect to the arbitrarily selected w and z axes. The first step in the combination of several ellipses is to transform the individual values of σ_x and σ_y to values along the w and z axes. Such a transformation will be shown to involve also a cross-product function ρ in addition to the new standard deviations σ_w and σ_z. The necessary transformation equations are the standard ones for the rotation of coordinate axes.

$$w = x \cos\theta - y \sin\theta$$

$$z = y \cos\theta + x \sin\theta$$

or equivalently,

$$x = w \cos\theta + z \sin\theta$$

$$y = z \cos\theta - w \sin\theta \quad .$$

The original distribution in terms of x and y is given by

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left[\frac{x^2}{\sigma_x^2} + \frac{y^2}{\sigma_y^2}\right]}$$

Substituting the transformed values for x and y, we obtain

$$f(w,z) = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left[\frac{(w\cos\theta + z\sin\theta)^2}{\sigma_x^2} + \frac{(z\cos\theta - w\sin\theta)^2}{\sigma_x^2}\right]}$$

Considering now only the exponent of the general expression of an elliptical bivariate distribution, which is

$$-\frac{1}{2(1-\rho^2)}\left[\frac{w^2}{\sigma_w^2} - \frac{2\rho wz}{\sigma_w\sigma_z} + \frac{z^2}{\sigma_z^2}\right] \quad ,$$

one may equate these two exponents and solve for σ_w, σ_z and ρ. First, multiply out the trigonometric terms in the first exponent

$$-\frac{1}{2}\left[\frac{w^2\cos^2\theta}{\sigma_x^2} + \frac{2wz\sin\theta\cos\theta}{\sigma_x^2} + \frac{z^2\sin^2\theta}{\sigma_x^2} + \frac{z^2\cos^2\theta}{\sigma_y^2} - \frac{2wz\sin\theta\cos\theta}{\sigma_y^2} + \frac{w^2\sin^2\theta}{\sigma_y^2}\right]$$

rearranging terms

$$-\frac{1}{2}\left\{w^2\left[\text{[illegible]} + \frac{\sin^2\theta}{\sigma_y^2}\right] - 2wz\left[\sin\theta\cos\theta\left(\frac{1}{\sigma_y^2} - \frac{1}{\sigma_x^2}\right)\right] + z^2\left[\frac{\sin^2\theta}{\sigma_x^2} + \frac{\cos^2}{\sigma_y^2}\right]\right\} \quad .$$

The bracketed coefficients in each term may be replaced by A, C, and B, respectively, to give a simplified expression

$$-\frac{1}{2}\left[w^2A + 2Cwz + z^2B\right] \quad .$$

This simplified expression may now be equated to the general expression of the exponent and solved for the desired functions by comparison of similar parts.

$$w^2A - 2Cwz + z^2B = \frac{1}{(1-\rho^2)}\left[\frac{w^2}{\sigma_w^2} - \frac{2\rho wz}{\sigma_w\sigma_z} + \frac{z^2}{\sigma_z^2}\right]$$

$$w^2A = \frac{1}{1-\rho^2}\left(\frac{w^2}{\sigma_w^2}\right)$$

$$\sigma_w^2 = \frac{1}{(1-\rho^2)A}$$

similarly

$$\sigma_z^2 = \frac{1}{(1-\rho^2)B}$$

$$-2Cwz = -\frac{2\rho wz}{\sigma_w \sigma_z} \quad .$$

Substituting the values obtained above for σ_w and σ_z

$$C = -\frac{\rho(1-\rho^2)\sqrt{AB}}{(1-\rho^2)}$$

$$\rho = \frac{C}{\sqrt{AB}} \quad .$$

Thus the formulas for σ_w, σ_z and ρ indicated in Section 3 have been derived. The subscript i used in Section 3 indicates the particular ellipse by number where $i = 1, 2, 3 \ldots n$.

Following the computation of the functions pertaining to each separate ellipse with respect to the w and z axes, it is then necessary to combine these to obtain the parameters of the final combined error ellipse. Since the final error is the sum of individual errors, then the moment-generating function of the final distribution is the product of the moment-generating functions of the individual distributions (since the distributions are independent). The moment-generating function for w_i, z_i is:

$$m_i(t_1, t_2) = e^{\frac{1}{2}\left(t_1^2\sigma_{w_i}^2 + 2\rho_i t_1 t_2 \sigma_{w_i}\sigma_{z_i} + t_2^2\sigma_{z_i}^2\right)} \quad .$$

Now

$$m(t_1, t_2) = \prod_{i=1}^{n} m_i(t_1, t_2) = e^{\frac{1}{2}\left[t_1^2\left(\sigma_{w_1}^2 + \sigma_{w_2}^2 + \ldots \sigma_{w_2}^2\right) + 2t_1t_2\left(\rho_1\sigma_{w_1}\sigma_{z_1} + \ldots \rho_n\sigma_{w_n}\sigma_{z_n}\right) + t_2^2\left(\sigma_{z_1}^2 + \sigma_{z_2}^2 + \ldots \sigma_{z_n}^2\right)\right]}$$

where $m(t_1, t_2)$ is the moment-generating function of the final distribution and

$$m(t_1, t_2) = e^{\frac{1}{2}\left(t_1 \sigma^2_{w_f} + 2\rho_f t_1 t_2 \sigma_{w_f} \sigma_{z_f} + t_2^2 \sigma^2_{z_f}\right)} .$$

Identifying similar parts, we have

$$\sigma^2_{w_f} = \sum_{i=1}^{n} \sigma^2_{w_i}$$

$$\sigma^2_{z_f} = \sum_{i=1}^{n} \sigma^2_{z_i}$$

$$\rho_f = \frac{1}{\sigma_z \sigma_w} \sum_{i=1}^{n} \rho_i \sigma_{w_i} \sigma_{z_i}$$

Having thus obtained the parameters of the final combined error ellipse with respect to the arbitrary w and z axes, it is further desirable to convert to the x and y axes of the final ellipse to remove the cross-product function ρ_f. Formulas for this transformation may be found in Hald, Ref. 8.

$$\sigma_{x_f} \sigma_{y_f} = \sigma_{w_f} \sigma_{z_f} \sqrt{1 - \rho_f^2} \qquad \text{(Hald 19.8.10)}$$

$$\sigma^2_{x_f} + \sigma^2_{y_f} = \sigma_{w_f} \sigma_{z_f} \qquad \text{(Hald 19.8.11)}$$

These equations are identical in form with (34) and (35) of Appendix 2. Thus the solutions may be given by analogy as

$$\sigma^2_{x_f} = \frac{1}{2}\left[\sigma^2_{w_f} + \sigma^2_{z_f} + \sqrt{\left(\sigma^2_{w_f} + \sigma^2_{z_f}\right)^2 - 4\sigma_{w_f}\sigma_{z_f}(1 - \rho^2)}\right]$$

$$\sigma^2_{y_f} = \frac{1}{2}\left[\sigma^2_{w_f} + \sigma^2_{z_f} - \sqrt{\left(\sigma^2_{w_f} + \sigma^2_{z_f}\right)^2 - 4\sigma_{w_f}\sigma_{z_f}(1 - \rho^2)}\right]$$

The relationship of the x axis of the final ellipse to the w and z axes is given by

$$\tan 2\theta_f = \frac{2\rho_f \sigma_{w_f} \sigma_{z_f}}{\sigma^2_{w_f} - \sigma^2_{z_f}} \qquad \text{(Hald 19.8.7)}$$

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